ASSESSMENT OF APPROXIMATE ERROR VALUES USED IN ASTRONOMICAL NAVIGATION FOR POSITIONING

Andrei POCORA\textsuperscript{1}
Sergiu LUPU\textsuperscript{2}
Elena Carmen LUPU\textsuperscript{2}

\textsuperscript{1} Assistant Professor eng. PhD student, "Mircea cel Batran" Naval Academy, Constanta, Romania andrei.pocora@anmb.ro
\textsuperscript{2} Lecturer PhD, "Ovidius" University, Constanta, Romania sergiu.lupu@anmb.ro

Abstract: Finding the vessel's position on the terrestrial sphere represents the most important activity carried out by the officer of the watch onboard, in order to ensure the safety of the vessel, crew and cargo. The ship's position can be determined by several methods such as: coastal observations, astronomical observations, radar or through data provided by satellite global positioning systems. The position of the vessel with astronomical observations, the officer of the watch uses a series of nautical tables and formulas of spherical trigonometry applied to the spherical triangle of position. The approximate values accuracy of the trigonometric functions used in computing can directly affect the position determined by astronomical observations. The purpose of this paper is to evaluate the errors of approximate values of the trigonometric functions used by the officer on watch for fix positioning with astronomical observations.

Introduction
According to Standards of Training and Watchkeeping Code (STCW), the Officer in Charge with Navigational Watch (OOW) is the master's representative on board the ship. He is primarily responsible at all times for the safe navigation of the ship for the prevailing circumstances and conditions until properly relieved.

In order to maintain a safe navigational watch, the navigational duties of the OOW has to execute the plan safely and to monitor the progress of the ship to that plan. For that, the OOW must know all the time the position of the ship by geographical coordinates.

The ship’s position can be determined by several methods such as: coastal observations, astronomical observations, radar or through data provided by satellite global positioning systems. Another duty of the OOW is to make periodic checks on the navigational equipment in use. He must check and record gyro and magnetic compass errors at least once a watch, where possible, and after any major course alteration [1-2]. The gyro and magnetic compass error can be determined using coastal or astronomical methods.

The theoretical elements of navigation
Practicing navigation involves performing a series of mathematical calculations that include the use of trigonometric functions. For the results to be high enough to trust it is required to comply with certain rules based on mathematical considerations and long experience of seafarers.

All calculations for solving problems of navigation must match the precision of the calculations. Increasing the accuracy of the calculations significantly above the precision of the observations does not imply improving the accuracy of the final result, but the increase in workload and time required to solve the problem. Executed navigation observations are characterized by a limited accuracy imposed by the precision of the navigation instruments used and by the precision of their execution. A series of values used in navigation are determined through calculus, characterized by a certain degree of approximation. It is very important to know from the beginning the approximation degree of the calculus (rounded, truncated values) and to maintain the precision of the observations, in order to maintain the precision of the final result.

Determining the geographical coordinates of the ship - latitude (Lat.) and longitude (Long.) - is performed according to the spherical coordinates of the stars at which the observations are made. Determination of the ship’s astronomical position can be achieved with simultaneous and successive observations.

The celestial point is determined at the intersection of two or three lines of position (LOP), according to the line of position theory [3-6]. The elements of the straight lines of position LOP (intercept and azimuth) are calculated for the same estimated point Ze (estimated latitude and longitude). This method is most applicable when the stars are visible at the same time, in positions that provide favorable conditions, namely:

- during twilight (when is visible the horizon and celestial bodies);
- during moonlight nights, depending on the horizon visibility;
- during daylight with the Sun using successive observations.

Determination of the errors of the gyro or magnetic compass with astronomical observations is made by comparing the azimuths and gyro or magnetic compass bearings of the observed stars.

Astronomical calculations are performed by applying trigonometric formulas in the spherical triangle position.

Calculating the intercept is made by using spherical trigonometric formula \( \sin \theta \). The azimuth angle (\( \theta \)) of a star can be achieved by using spherical trigonometry from formulas \( \sin \theta \), \( \cot \theta \) or with A.B.C. tables using Hydrographic Direction Tables (D.H.-90) or Norie’s Nautical Tables.

The spherical triangle of position arises through the intersection of three great circles:

- the observer’s celestial meridian;
- the vertical circle of the star;
- the hour circle of the star.

The elements of a spherical triangle are:

- the triangle’s peaks;
- the triangle’s sides;
- the triangle’s angles.

The peaks of spherical triangle are:
- the zenith (Z);
- the high celestial pole No (S);
- the star A.

Spherical triangle’s sides are great arcs resulted by combining horizontal and equatorial coordinates at the intersection of the three great circles:
- the colatitude: Col. = 90° - Lat.;
- the zenith distance: z = 90° - Hc;
- the polar distance: = 90° - Dec.;

Spherical triangle’s peaks are as follows:
- the zenith angle (Zn);
- the meridian angle (t);
- the parallactic angle A.

The quadrantal azimuth angle (Zc) is expressed as
\[ Zc = 90° \pm n \] where:
- Calculated altitude (Hc) is an angular distance above the horizon measured along a vertical circle, from 0° at the horizon to 90° at the zenith. It is positive in the visible hemisphere and negative in the invisible hemisphere;
- Latitude (Lat.) is the angle between a line in the direction of a station and the plane of the equator with values from 0° to 90°, north or south of the celestial equator;
- Declination (Dec.) is an angular distance north or south of the celestial equator. It is measured along an hour circle, from 0° at the celestial equator to 90° at the celestial poles. It is positive in the visible hemisphere and negative in the invisible hemisphere;
- Meridian angle (t) is an angular distance west or east of the local celestial meridian in eitheran easterly or westerly direction; values from 0° to 180°.
- Quadrantal azimuth angle (Zc) is an arc of the horizon measured either clockwise or counterclockwise with values from 0° to 90°, starting at the north or south point of the horizon.
- Semicircular azimuth angle (Zs) is an arc of the horizon measured either clockwise or counterclockwise with values from 0° to 180°, starting at the north point of the horizon in north latitude and the south point of the horizon in south latitude.

The azimuth is determined from the quadrantal or semicircular zenith angle.

\[ \sin(Zc) = \sin(Zn) \pm \sin(Lat.) \cos(Dec.) \] (1)
\[ \tan(Zc) = \frac{\sin(Zn)}{\cos(Dec.) \cos(Lat.)} \] (2)
\[ \cot(Zc) = \frac{\sin(Dec.) \cos(Lat.)}{\sin(Lat.)} \] (3)

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- Calculated altitude (Hc) is an angular distance above the horizon measured along a vertical circle, from 0° at the horizon to 90° at the zenith. It is positive in the visible hemisphere and negative in the invisible hemisphere;
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The azimuth (Zn) is an arc of the horizon measured clockwise starting from the north point on the horizon. It takes values from 0° to 360°.

The azimuth is determined from the quadrantal or semicircular azimuth angle. The quadrantal azimuth angle (Zc) it is expressed as follow:

\[ Zc = NEa^o \rightarrow Zn = a^o \] \[ Zc = NWa^o \rightarrow Zn = 360° - a^o \] \[ Zc = SEa^o \rightarrow Zn = 180° - a^o \]
The values of logarithms of trigonometrical functions can be found in tables: in D.H.-90 or Norie’s Nautical Tables. Its is expressed as numerical values with five decimal places. The second method is based on direct application of formula (1-3). The D.H.-90 or Norie’s Nautical Tables contains tables with the values of trigonometrical functions, Its is expressed as numerical values with five decimal places.

**The mathematical evaluation**

Mathematical processing of the observations results achieved with approximate numbers and the accuracy of the calculations must always match the precision of observations. The calculations must be performed using formulas tested in practice and ensure speed in obtaining the final output. Calculation results should not contain errors and omissions. Therefore, the results must be provided with caution for intermediate operations. Calculations must be arranged types of rotational calculation and to obtain the necessary promptness in work and avoid mistakes and omissions.

The accuracy of organizing operations and writing figures are of special importance given that, usually onboard ship, calculations are performed by one person and often, time does not allow to be repeated for verification.

The approximate value of a given quantity has as a basis:
- all observations made, are characterized by limited accuracy, meaning that each of the observations represents a close value to the true one, depending on the precision of the observation;
- not all the values can be observed (measured) directly, meaning that their value do not result from observations but from mathematical relations and calculus characterized by a certain degree of approximation;
- in calculus, arithmetic operations are always present and accompanied by a certain.

Thus, along with the approximate value obtained from observations, in calculations inherent approximation appear due to arithmetic operations. This is the reason why it is very important to know from the beginning the degree of approximation for calculations. Thus the approximation from calculations (rounding) should not reduce the precision of the observations and so, the precision of the final result will not be reduced.

The notion of approximate error entails the existence of errors that characterize the approximate values. These can be absolute or relative [7].

The absolute error of an approximate value is given by the difference between the precise value and the approximate value:

\[ \Delta = X - A \]  

where:
- \( X \) – represent the precise value;
- \( A \) – represent the approximate value.

The relative error (\( \delta \)) of the approximate value is given by the ratio between the absolute error and the precise value:

\[ \delta = \frac{\Delta}{A} \]  

Thus, relative of to the relative error, the relation for the precise value is:

\[ X = A \left( 1 + \frac{\Delta}{A} \right) = A(1 + \delta) \]  

The relative errors are usually expressed in percents (\( \% \)):

\[ \% = \frac{\delta}{A} \times 100 \]  

For approximate values in which a large number of decimals is known, the sign of the error can be determined. This can not be said for the values obtained from observations. If a approximate value of the absolute error can be determined, the sign of the error can not be determined. The absolute errors and concretely numbers and have the same value as the value of the observation. The relative errors are abstract numbers without having any unit of measure.

The absolute error of the sum/difference between the approximate numbers is equal to the sum/difference between the absolute errors of the numbers of which this sum/difference is formed.

Thus, the absolute error of multiplications is the difference between the exact values and the multiplication of approximate values:

\[ \Delta = \frac{\Delta_1 \Delta_2}{A_1^2} \]  

**Assessment of approximate error values**

May the ship be anchored in the estimated position: Lat.= 44°32’ N, Long.= 029°57’ E on april 28, 2015. To determine the fix position of the ship, the officer of the watch use two sextant star sights: Markab and Rasalhague. At the chronometer time (C) C=03°47’30’’ the officer of the watch observes Markab and notes the sextant altitude (Hs) Hs=43°17’.

At the chronometer time (C) C=03°51’34’’ the officer of the watch observes Rasalhague and notes the sextant altitude (Hs) Hs=43°26’.

First of all, the he must determine the LOP. For that, he will determine:

1. the value of the Universal Time (UT) for the moment of the sextant star sight. It is determined from the chronometer time (C) and the chronometer error (CE);
2. the values of the meridian angle (\( \theta_{\text{MW}} \)) and the declination (Dec.) of the star at the UT of sight using astronomical ephemeris (Brown’s Nautical Almanac or The Nautical Almanac);
3. the value of the calculated altitude (Hc) using the mathematical formula (1);
4. the value of the azimuth (Zn) using the mathematical formula (3);
5. the value of the true altitude (Ho) from the sextant altitude sight.
6. the intercept (p) which is the difference between the true altitude and calculated altitude.

After these steps, the officer of the watch will plot on the paper chart that two lines of position to determin the fix position.

Knowing the values of the chronometer error (CE=+1’07’), the sextant index (\( \theta_{\text{MW}}=-1’\)), and the height of eye (h=12m), the officer of the watch will determine the lines of positions.

For Markab:
1. UT=03°48’37’’
2. \( \theta_{\text{MW}} = 43°20’ \)
3. Dec.=N15°17’
4. Hc= 43°09’
5. Zn=114°.
6. Ho=43°10’
7. p=+1’.0

For Rasalhague:
1. UT=03°52’41’’
2. \( \theta_{\text{MW}} = 40°00’ \)
3. Dec.=N12°33’
4. Hc= 43°15’
5. Zn=239°
6. Ho=43°17’
7. p=+2’.0

The calculated altitude of the Markab was determined using the precise values of the trigonometric functions (Tab. 1).
The accuracy of the calculated altitude is expressed in degrees, minutes, and decimal of minutes. Thus, the value of calculated altitude is $H_c = 43°09'11'',$ because 5.97'' (arcseconds) represents onethirtieth of a minute of arc.

Using only five decimals for trigonometric functions according to the current practice of navigation it was determined the calculated altitude, absolute and relative errors (Tab. 2).

<table>
<thead>
<tr>
<th>The Trigonometric Functions</th>
<th>Value with 5 decimals</th>
<th>The absolute error</th>
<th>The relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin(Lat.)</td>
<td>0.70132</td>
<td>40.979 x 10^-7</td>
<td>0.00058</td>
</tr>
<tr>
<td>cos(Lat.)</td>
<td>0.71284</td>
<td>25.559 x 10^-7</td>
<td>0.00036</td>
</tr>
<tr>
<td>sin(Dec.)</td>
<td>0.26362</td>
<td>5.203 x 10^-7</td>
<td>0.00020</td>
</tr>
<tr>
<td>cos(Dec.)</td>
<td>0.96463</td>
<td>-35.327 x 10^-7</td>
<td>-0.00037</td>
</tr>
<tr>
<td>cos(t.)</td>
<td>0.72575</td>
<td>47.052 x 10^-7</td>
<td>0.00065</td>
</tr>
<tr>
<td>a</td>
<td>0.18488</td>
<td>34.236 x 10^-7</td>
<td>0.00185</td>
</tr>
<tr>
<td>b</td>
<td>0.49905</td>
<td>-16.170 x 10^-7</td>
<td>-0.00032</td>
</tr>
<tr>
<td>sin(Hc)</td>
<td>0.68393</td>
<td>-18.066 x 10^-7</td>
<td>-0.00026</td>
</tr>
<tr>
<td>Hc</td>
<td>43.15166</td>
<td>-24.039 x 10^-7</td>
<td>-0.00001</td>
</tr>
</tbody>
</table>

Using different numbers of decimals for the trigonometric functions the absolute error and the error in position in meters for the calculated altitude was determined (Tab. 3).

<table>
<thead>
<tr>
<th>Numbers of decimal places</th>
<th>The Hc value</th>
<th>The absolute error</th>
<th>The error in position</th>
</tr>
</thead>
<tbody>
<tr>
<td>Precisevalue</td>
<td>43°09'5.97''</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>43°09'5.98''</td>
<td>-24.039 x 10^-7</td>
<td>0.3 m</td>
</tr>
<tr>
<td>4</td>
<td>43°09'6.12''</td>
<td>-42.403 x 10^-8</td>
<td>4.6 m</td>
</tr>
<tr>
<td>3</td>
<td>43°09'7.2''</td>
<td>-34.240 x 10^-8</td>
<td>3.9 m</td>
</tr>
<tr>
<td>2</td>
<td>43°09'9''</td>
<td>16.575 x 10^-8</td>
<td>-184.2 m</td>
</tr>
<tr>
<td>1</td>
<td>43°12''</td>
<td>-48.342 x 10^-7</td>
<td>5371.7 m</td>
</tr>
</tbody>
</table>

It can be seen from the graph (Fig. 4) that the error of calculated altitude (Hc) increases exponentially when twodecimals are used for its determination.

**Conclusions**
In astronomical navigation problems, in order to determine the calculated altitude or azimuth angle, the officer on watch can use a series of nautical publications or a scientific calculator. Considering the precision of the calculated altitude that the officer of the watch operates on board the ship, the values of the trigonometric functions can be at least three decimals. To determine the azimuth angle, the values of the trigonometric functions can be at least two decimals.

Bibliography

[1] ***IMO, STCW-Standards of Training, Certification and Watchkeeping for Seafarers