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NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING INVOLVING HIGHER-ORDER GENERALIZED (*F*,□ *b*)-CONVEXITY

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Abstract: *We define higher-orde (F,□ ,b)-convexity and generalized higher-order (F,□ ,b)-convexity. We consider a nondifferentiable multiobjective programming problem involving support functions and present a higher-order dual model for this problem and we prove some duality results under appropriate higher-order (F,□ ,b)-convexity conditions.*
Keywords: *Higher-order duality, nondifferentiable multiobjective programming, higher-order (F,□ ,b)-(pseudo-, quasi-) convexity.*

1. INTRODUCTION

For nonlinear programming problems, a number of duals have been suggested among which the Wolfe dual [14] is well known. While studying duality under generalized convexity, Mond and Weir [18] proposed a number of different duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity / quasi-convexity assumptions.

Taking motivation from Bazaraa and Goode [1] and Hanson and Mond [9], Nanda and Das [20] attempted to extend the results of Mond and Weir [18] to cone domains with appropriate pseudo-invexity and quasi-invexity assumptions on objective and constraint functions. However, certain shortcomings were pointed out in the work of Nanda and Das [20] and appropriate modifications were suggested for studying duality under pseudo-invexity assumptions in Chandra and Abha [4].

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used [11, 13, 17, 24]. Mangasarian [13] considered a nonlinear programming and discussed second order duality under inclusion condition. Mond [17] was the first to present second order convexity. He also gave in [17] simpler conditions than Mangasarian using a generalized form of convexity which was later called second order convexity by Mahajan [12] and bonvexity by Bector and Chandra [3]. Later, Jeyakumar [11] and Yang [24] discussed also second order Mangasarian type dual formulation under □-convexity and generalized representation conditions respectively. In [28] Zhang and Mond established some duality theorems for second-order duality in nonlinear programming under second order ***B***-invexity or generalized second-order ***B***-invexity, defined in their paper. In [2, 18, 8] was shown that second order duality can be useful computationally, since one may obtain better lower bounds for the primal problem than otherwise. The case of some optimization problems that involve *n*-set functions was studied by Preda [22].

Recently, Yang, Yang, Teo and Hou [27] proposed four second-order dual models for nonlinear programming and established some duality results under generalized second-order ***F***-convexity assumptions. In [16] Mishra and Rueda generalized Zhang's Mangasarian type and Mond-Weir type higher-order duality [28] to higher-order type I functions. Yang, Teo and Yang [26] extended this results to a class of nondifferentiable multiobjective programming problems. They also presented an unified higher-order dual model for nondifferentiable multiobjective programs, where every component of the objective function contains a support function of a compact convex set.

In this paper, we define in section 2 the higher-order (*F*,□ ,*b*)-convexity and generalized higher-order (*F*,□ ,*b*)-convexity. In section 3 we consider a class of nondifferentiable programming problems and for the dual model defined by Yang, Teo and Yang [26], we prove some weak duality results.

2. DEFINITIONS AND SOME PRELIMINARY RESULTS

We denote by $\mathbf{R}^n$ the *n*-dimensional Euclidean space, and by $\mathbf{R}^n_+$ the nonnegative orthant of $\mathbf{R}^n$.

For any vectors $x \in \mathbf{R}^n, y \in \mathbf{R}^n,$ we denote: $x^{\mathrm{T}} y = \sum_{i=1}^{n} x_i y_i.$

Let $C \subset \mathbf{R}^n$ be a compact convex set. The support function of *C* is defined by

$$s(x \mid C) = \max\left\{x^{\mathrm{T}} y \mid y \in C\right\}$$

and being convex and everywhere finite, it has a subdifferential [30], that is, there exists $z \in \mathbf{R}^n$ such that

$$s(y \mid C) \geq s(x \mid C) + z^{\mathrm{T}}(y - x) \quad \text{for all } y \in C.$$

The subdifferential of $s(x \mid C)$ is given by

$$\partial s(x \mid C) = \left\{z \in C \mid z^{\mathrm{T}} x = s(x \mid C)\right\}.$$

For any set $D \subset \mathbf{R}^n,$ the normal cone to *D* at a point $x \in D$ is defined by

$$N_D(x) = \left\{y \in \mathbf{R}^n \mid y^{\mathrm{T}}(z - x) \leq 0, \quad \text{for all } z \in D\right\}.$$

For a compact convex set *C* we obviously have: $y \in N_C(x)$ if and only if $s(y \mid C) = x^{\mathrm{T}} y,$ or equivalently, if $x \in \partial s(y \mid C).$

Let us consider $H : \mathbf{R}^n \to \mathbf{R}^p$, $G : \mathbf{R}^n \to \mathbf{R}^q$, and $X \subset \mathbf{R}^n$. We define the following multiobjective programming problem:

$$\begin{cases} \text{minimize } H(x) \\ \text{subject to: } G(x) \ge 0, \quad x \in X \end{cases} \qquad \text{(P)}$$

We denote the set of feasible solutions of (P) by $\mathcal{P}$, that is:

$$\mathcal{P} = \{ x \in X \mid G(x) \ge 0 \}.$$

Definition 2.1 A vector $\overline{x} \in \mathcal{P}$ is an efficient solution of (P) if there exists no other $x \in \mathcal{P}$ such that $H(\overline{x}) - H(x) \in \mathbf{R}_+^p \setminus \{0\}$, that is, $H_i(x) \le H_i(\overline{x})$ for all $i \in \{1, \dots, p\}$, and for at least one $j \in \{1, \dots, p\}$ we have $H_j(x) < H_j(\overline{x})$. $\overline{x} \in \mathcal{P}$ is said to be a weak efficient solution of (P) if there exists no $x \in \mathcal{P}$ such that for all $i \in \{1, \dots, p\}$, $H_i(x) < H_i(\overline{x})$.

Definition 2.2 An efficient solution $\overline{x} \in \mathcal{P}$ of (P) is properly efficient, if there exists a positive number *M* with the property that, whenever $H_i(x) < H_i(\overline{x})$ for $x \in \mathcal{P}$ and $i \in \{1, \dots, p\}$, there exists some $j \in \{1, \dots, p\}$ such that $H_j(x) > H_j(\overline{x})$ and $\dfrac{H_i(\overline{x}) - H_i(x)}{H_j(x) - H_j(\overline{x})} \le M$.

For a real-valued twice differentiable function $\psi(x, y)$ defined on an open set in $\mathbf{R}^p \times \mathbf{R}^q$, we denote by $\nabla_x \psi(\overline{x}, \overline{y})$ the gradient vector of ψ with respect to *x* at $(\overline{x}, \overline{y})$, and by $\nabla_{xx} \psi(\overline{x}, \overline{y})$ the Hessian matrix with respect to *x* at $(\overline{x}, \overline{y})$. Similarly we may define $\nabla_y \psi(\overline{x}, \overline{y})$, $\nabla_{xy} \psi(\overline{x}, \overline{y})$ and $\nabla_{yy} \psi(\overline{x}, \overline{y})$.

The following lemma will be used in the next sections.

Lemma 2.1 ([7]) *If $\overline{x} \in \mathcal{P}$ is a properly efficient solution of (P), there exist $\alpha \in \mathbf{R}_+^p \setminus \{0\}$ and $\beta \in \mathbf{R}_+^q \setminus \{0\}$ such that*

$$\sum_{i=1}^{p} \alpha_i \nabla_x H_i(x) - \sum_{j=1}^{q} \beta_j \nabla_x G_j(x) = 0.$$

Definition 2.3 A function $F : X \times X \times \mathbf{R}^n \to \mathbf{R}$ (where $X \subseteq \mathbf{R}^n$) is sublinear with respect to the third variable if for all $(x, y) \in X \times X$, we have:

(i) $F(x, y; a_1 + a_2) \le F(x, y; a_1) + F(x, y; a_2)$ for all $a_1, a_2 \in \mathbf{R}^n$,

(ii) $F(x, y; ra) = rF(x, y; a)$, for all $r \in \mathbf{R}_+$, $a \in \mathbf{R}^n$.

Let us consider the functions $b : X \times X \to \mathbf{R}_+$, $d : X \times X \to \mathbf{R}_+$, and the number $\rho \in \mathbf{R}$. Further, we suppose that $F : X \times X \times \mathbf{R}^n \to \mathbf{R}$ (where $X \subseteq \mathbf{R}^n$) is sublinear with respect to the third variable and that $\phi : X \to \mathbf{R}$ and $h : X \times \mathbf{R}^n \to \mathbf{R}$ are differentiable functions.

We introduce in the subsequent definition the class of higher-order (F, ρ, b)-convexity.

Definition 2.4

- We say that φ is higher-order (F, ρ, b)-convex at $u \in X$ with respect to *h*, if for all $(x, y) \in X \times \mathbf{R}^n$ we have

$$b(x,u)\big(\varphi(x) - \varphi(u)\big) \ge F\big(x, u; \nabla_x \varphi(u) + \nabla_y h(u, y)\big) + \\ + b(x,u)\big(h(u,y) - y^{\mathrm{T}}\left[\nabla_y h(u,y)\right]\big) + \rho d(x,u)$$

- We say that φ is higher-order (F, ρ, b)-pseudo-convex at $u \in X$ with respect to *h*, if for all $(x, y) \in X \times \mathbf{R}^n$ we have

$$F\big(x, u; \nabla_x \varphi(u) + \nabla_y h(u, y)\big) \ge -\rho d(x,u) \\ \Rightarrow b(x,u)\big(\varphi(x) - \varphi(u)\big) \ge b(x,u)\big(h(u,y) - y^{\mathrm{T}}\left[\nabla_y h(u,y)\right]\big)$$

- We say that φ is higher-order (F, ρ, b)-quasi-convexe at $u \in X$ with respect to *h*, if for all $(x, y) \in X \times \mathbf{R}^n$ we have

$$b(x,u)\big(\varphi(x)-\varphi(u)\big)\le b(x,u)\Big(h(u,y)-y^{\mathrm{T}}\big[\nabla_y h(u,y)\big]\Big)$$
$$\Rightarrow F\big(x,u;\nabla_x\varphi(u)+\nabla_y h(u,y)\big)\le -\rho d(x,u)$$

- If φ is higher-order (F,ρ,b)-convex (-pseudo-convex, -quasi-convex) at each point $u\in X$ with respect to the same function *h*, then φ is said to be higher-order (F,ρ,b)-convex (-pseudo-convex, -quasi-convex) on *X* with respect to *h*.
- If $-\varphi$ is higher-order (F,ρ,b)-convex (-pseudo-convex, -quasi-convex) at $u\in X$ with respect to *h*, then φ is said to be higher-order (F,ρ,b)-concave (-pseudo-convave, -quasi- con- cave) at $u\in X$ with respect to *h*.

Remarks

1) When $\rho=0$ and $b\equiv 1,$ the above definition reduce to Definition 4 of Chen [5].
2) When $\rho=0,\ b\equiv 1,\ h(u,y)=y^{\mathrm{T}}\nabla_{xx}\varphi(u)y/2,$ and $F(x,u;a)=\eta(x,u)^{\mathrm{T}}a,$ where $\eta : X\times X\to\mathbf{R}^n,$ the higher-order (F,ρ,b)-convexity (-pseudo- convexity, -quasi-convexity) reduces to η-bonvexity (-pseudo-bonvexity, -quasi- bonvexity) in [7, 21], or it reduces to the second-order *F* (pseudo-, quasi-) invexity in [10].
3) When $\rho=0,\quad b\equiv 1,\quad h(u,y)=-y^{\mathrm{T}}\nabla_x\varphi(u)+\psi(u,y),$ and $F(x,u;a)=\alpha(x,u)a^{\mathrm{T}}\eta(x,u),$ where $\alpha : X\times X\to\mathbf{R}_+\setminus\{0\},\quad \eta : X\times X\to\mathbf{R}^n$ are positive functions, and $\psi : X\times\mathbf{R}^n\to\mathbf{R}$ is a differentiable function, the higher-order (F,ρ,b)-convex (-pseudo-convex, -quasi-convex) function becomes the higher-order (pseudo-, quasi-) type I function in [15, 19].

3. HIGHER-ORDER DUALITY INVOLVING NONDIFFERENTIABLE FUNCTIONS

We consider in this section the differentiable functions

$$f=\left(f_1,\ldots,f_p\right)^{\mathrm{T}} : \mathbf{R}^n\to\mathbf{R}^p,$$
$$g=\left(g_1,\ldots,g_m\right)^{\mathrm{T}} : \mathbf{R}^n\to\mathbf{R}^m,$$
$$k=\left(k_1,\ldots,k_m\right)^{\mathrm{T}} : \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^m,$$
$$h:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R},$$

the compact convex sets $C_i\subset\mathbf{R}^n,\ i\in P=\{1,2,\ldots,p\},$ and the open subset $D\subset\mathbf{R}^n.$ We also denote $M=\{1,2,\ldots,m\}$ and consider a partition of the index set *M* defined by $I_\alpha\subset M,\ \alpha=0,1,\ldots,\mu,$ with $\bigcup_{\alpha=0}^{\mu} I_\alpha=M$ and $I_\alpha\cap I_\beta=\varnothing$ for any $\alpha\neq\beta$.

We define the following nondifferentiable multiobjective programming problem:

$$\begin{aligned}&\text{minimize: }\left\{f_1(x)+s(x\,|\,C_1),\ldots,f_p(x)+s(x\,|\,C_p)\right\}\\&\text{subject to: } g(x)\ge 0,\ x\in D.\end{aligned}\qquad\text{(NMP)}$$

To this problem, we associate the following general dual

$$\left.\begin{aligned}\text{maximize: }\ \{\ & f_1(u)+h_1(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi h_1(u,\pi)+u^{\mathrm{T}}w_1-\\ &-\sum_{\tau\in I_0}\left[y_\tau g_\tau(u)+y_\tau k_\tau(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi y_\tau k_\tau(u,\pi)\right],\\ &\cdots\\ & f_p(u)+h_p(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi h_p(u,\pi)+u^{\mathrm{T}}w_p-\\ &-\sum_{\tau\in I_0}\left[y_\tau g_\tau(u)+y_\tau k_\tau(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi y_\tau k_\tau(u,\pi)\right]\ \},\end{aligned}\right\}\quad\text{(NMD)}$$

subject to:

$$\lambda^{\mathrm{T}}\nabla_\pi h(u,\pi)+\sum_{i=1}^{p}\lambda_i w_i=\nabla_\pi\left(y^{\mathrm{T}}k(u,\pi)\right)\qquad(3.1)$$

$$\sum_{\tau\in I_\alpha}\left[y_\tau g_\tau(u)+y_\tau k_\tau(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi\left(y_\tau k_\tau(u,\pi)\right)\right]\le 0,\ \ \alpha=1,\dots,\mu, \qquad (3.2)$$

$$y\ge 0, \quad (3.3)$$

$$w_i\in C_i,\ i\in P,\ \ \lambda\in\Lambda^+, \qquad (3.4)$$

where $\Lambda^+=\left\{\lambda\in\mathbf{R}^p\ \middle|\ \lambda>0,\ \sum_{i=1}^{p}\lambda_i=1\right\}$. In the sequel we denote $w=(w_1,\dots,w_p)$.

Weak Duality

Theorem 3.1 *Let x be feasible for (NMP) and let* (u,λ,w,y,π) *be feasible for (NMD). Suppose that for all feasible* (x,u,y,w,π) *there exist a sublinear function* $F\ :\ \mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ *such that*

$$\sum_{\tau\in I_\alpha}\left[-y_\tau g_\tau(u)+y_\tau k_\tau(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi\left(y_\tau k_\tau(u,\pi)\right)\right]\le 0\Rightarrow$$
$$\Rightarrow F\left(x,u,-\sum_{\tau\in I\alpha}y_\tau\nabla_\pi k_\tau(u,\pi)\right)\le-\beta_\alpha d(x,y),\ \ \alpha=1,\dots,\mu. \qquad (3.5)$$

Furthermore, it is assumed that one of the following three conditions holds:

(a) *For* $i\in P$,

$$F\left(x,u,\nabla_\pi h_i(u,\pi)+w_i-\sum_{\tau\in I_0}y_\tau\nabla_\pi k_\tau(u,\pi)\right)\ge-\rho_i d(x,u)\Rightarrow$$
$$\Rightarrow f_i(x)+x^{\mathrm{T}}w_i-\left(f_i(u)+u^{\mathrm{T}}w_i+h_i(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi h_i(u,\pi)\right)+$$
$$+\sum_{\tau\in I_0}\left[y_\tau g_\tau(u)+y_\tau k_\tau(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi y_\tau k_\tau(u,\pi)\right]\ge 0\ ;$$

$$f_i(x)+x^{\mathrm{T}}w_i-\left(f_i(u)+u^{\mathrm{T}}w_i+h_i(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi h_i(u,\pi)\right)+$$
$$+\sum_{\tau\in I_0}\left[y_\tau g_\tau(u)+y_\tau k_\tau(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi y_\tau k_\tau(u,\pi)\right]\le 0\Rightarrow$$
$$\Rightarrow F\left(x,u,\nabla_\pi h_i(u,\pi)+w_i-\sum_{\tau\in I_0}y_\tau\nabla_\pi k_\tau(u,\pi)\right)\le-\rho_i d(x,u)$$

and $\quad\sum_{\alpha=1}^{\mu}\beta_\alpha+\sum_{i=1}^{p}\lambda_i\rho_i\ge 0$;

(b) *There exists* $j\in P$ *such that*

$$F\left(x,u,\nabla_\pi h_j(u,\pi)+w_j-\sum_{\tau\in I_0}y_\tau\nabla_\pi k_\tau(u,\pi)\right)\ge-\rho_j d(x,u)\Rightarrow$$
$$\Rightarrow f_j(x)+x^{\mathrm{T}}w_j-\left(f_j(u)+u^{\mathrm{T}}w_j+h_j(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi h_j(u,\pi)\right)+$$
$$+\sum_{\tau\in I_0}\left[y_\tau g_\tau(u)+y_\tau k_\tau(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi y_\tau k_\tau(u,\pi)\right]\ge 0\ ;$$

while for all $i\in P$,

$$f_i(x)+x^{\mathrm{T}}w_i-\left(f_i(u)+u^{\mathrm{T}}w_i+h_i(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi h_i(u,\pi)\right)+$$
$$+\sum_{\tau\in I_0}\left[y_\tau g_\tau(u)+y_\tau k_\tau(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi y_\tau k_\tau(u,\pi)\right]\le 0\Rightarrow$$
$$\Rightarrow F\left(x,u,\nabla_\pi h_i(u,\pi)+w_i-\sum_{\tau\in I_0}y_\tau\nabla_\pi k_\tau(u,\pi)\right)\le-\rho_i d(x,u)$$

and $\quad\sum_{\alpha=1}^{\mu}\beta_\alpha+\sum_{i=1}^{p}\lambda_i\rho_i\ge 0$;

(c)

$$F\left(x,u,\lambda^{\mathrm{T}}\nabla_{\pi}h_i\left(u,\pi\right)+\sum_{i=1}^{p}\lambda_i w_i-\nabla_{\pi}y^{\mathrm{T}}k\left(u,\pi\right)\right)\geq-\rho d\left(x,u\right)\Rightarrow$$

$$\Rightarrow\lambda^{\mathrm{T}}f\left(x\right)+x^{\mathrm{T}}\sum_{i=1}^{p}\lambda_i w_i\geq\left(\lambda^{\mathrm{T}}f\left(u\right)+u^{\mathrm{T}}\sum_{i=1}^{p}\lambda_i w_i-y^{\mathrm{T}}g\left(u\right)\right)+$$

$$+\lambda^{\mathrm{T}}h\left(u,\pi\right)-y^{\mathrm{T}}k\left(u,\pi\right)-\pi^{\mathrm{T}}\left[\lambda^{\mathrm{T}}\nabla_{\pi}h\left(u,\pi\right)-y^{\mathrm{T}}\nabla_{\pi}k\left(u,\pi\right)\right]$$

and $\displaystyle\sum_{\alpha=1}^{\mu}\beta_\alpha+\rho\geq 0$

Then, the following relations cannot hold simultaneously:

$$\begin{aligned}&\text{for all } i\in P,\quad f_i\left(x\right)+s\left(x\,|\,C_i\right)\leq\\&\leq f_i\left(u\right)+u^{\mathrm{T}}w_i+h_i\left(u,\pi\right)-\pi^{\mathrm{T}}\nabla_{\pi}h_i\left(u,\pi\right)-\\&-\sum_{\tau\in I_0}\left[y_\tau g_\tau\left(u\right)+y_\tau k_\tau\left(u,\pi\right)-\pi^{\mathrm{T}}\nabla_{\pi}y_\tau k_\tau\left(u,\pi\right)\right],\end{aligned}\qquad(3.6)$$

and

$$\begin{aligned}&\text{for some } j\in P,\quad f_j\left(x\right)+s\left(x\,|\,C_j\right)<\\&< f_j\left(u\right)+u^{\mathrm{T}}w_j+h_j\left(u,\pi\right)-\pi^{\mathrm{T}}\nabla_{\pi}h_j\left(u,\pi\right)-\\&-\sum_{\tau\in I_0}\left[y_\tau g_\tau\left(u\right)+y_\tau k_\tau\left(u,\pi\right)-\pi^{\mathrm{T}}\nabla_{\pi}y_\tau k_\tau\left(u,\pi\right)\right].\end{aligned}\qquad(3.7)$$

Proof. Since x is feasible for (NMP) and $\left(u,\lambda,w,y,\pi\right)$ is feasible for (NMD), from (3.5) and the sublinearity of F, it follows that

$$F\left(x,u,-\sum_{\tau\in M\setminus I_0}y_\tau\nabla_{\pi}k_\tau\left(u,\pi\right)\right)\leq-\sum_{\alpha=1}^{\mu}\beta_\alpha d\left(x,y\right)\qquad(3.8)$$

From (3.1), (3.8) and the sublinearity of F, we obtain

$$F\left(x,u,\lambda^{\mathrm{T}}\nabla_{\pi}h\left(u,\pi\right)+\sum_{i=1}^{p}\lambda_i w_i-\sum_{\tau\in I_0}y_\tau\nabla_{\pi}k_\tau\left(u,\pi\right)\right)\geq\sum_{\alpha=1}^{\mu}\beta_\alpha d\left(x,y\right).\qquad(3.9)$$

Now we suppose on the contrary that (3.6) and (3.7) hold. Since $x^{\mathrm{T}}w_i\leq s\left(x\,|\,C_i\right),\ i\in P$, we have

$$\begin{aligned}&f_i\left(x\right)+x^{\mathrm{T}}w_i\leq f_i\left(x\right)+s\left(x\,|\,C_i\right)\leq\\&\leq f_i\left(u\right)+u^{\mathrm{T}}w_i+h_i\left(u,\pi\right)-\pi^{\mathrm{T}}\nabla_{\pi}h_i\left(u,\pi\right)-\\&-\sum_{\tau\in I_0}\left[y_\tau g_\tau\left(u\right)+y_\tau k_\tau\left(u,\pi\right)-\pi^{\mathrm{T}}\nabla_{\pi}y_\tau k_\tau\left(u,\pi\right)\right],\quad\forall\, i\in P,\end{aligned}\qquad(3.10)$$

and

$$\begin{aligned}&f_j\left(x\right)+x^{\mathrm{T}}w_j\leq f_j\left(x\right)+s\left(x\,|\,C_j\right)<\\&< f_j\left(u\right)+u^{\mathrm{T}}w_j+h_j\left(u,\pi\right)-\pi^{\mathrm{T}}\nabla_{\pi}h_j\left(u,\pi\right)-\\&-\sum_{\tau\in I_0}\left[y_\tau g_\tau\left(u\right)+y_\tau k_\tau\left(u,\pi\right)-\pi^{\mathrm{T}}\nabla_{\pi}y_\tau k_\tau\left(u,\pi\right)\right],\quad\text{for some } j\in P.\end{aligned}\qquad(3.11)$$

If case (a) is satisfied, then we obtain

$$F\left(x,u,\nabla_{\pi}h_i\left(u,\pi\right)+w_i-\sum_{\tau\in I_0}y_\tau\nabla_{\pi}k_\tau\left(u,\pi\right)\right)\leq-\rho_i d\left(x,u\right),\quad\forall\, i\in P,\qquad(3{,}12)$$

and

$$F\left(x,u,\nabla_{\pi}h_j\left(u,\pi\right)+w_j-\sum_{\tau\in I_0}y_\tau\nabla_{\pi}k_\tau\left(u,\pi\right)\right)<-\rho_j d\left(x,u\right),\ \text{for some } j\in P.\qquad(3.13)$$

Since $\lambda\in\Lambda^{+}$, it follows from (3.12), (3.13) and the sublinearity of F that

$$F\left(x,u,\sum_{i=1}^{p}\lambda_i\left(\nabla_\pi h_i(u,\pi)+w_i-\sum_{\tau\in I_0}y_\tau\nabla_\pi k_\tau(u,\pi)\right)\right)<-\sum_{i=1}^{p}\lambda_i\rho_i d(x,u). \qquad (3.14)$$

Since $\sum_{\alpha=1}^{\mu}\beta_\alpha+\sum_{i=1}^{p}\lambda_i\rho_i\geq 0$, from (3.14) we get

$$F\left(x,u,\sum_{i=1}^{p}\lambda_i\left(\nabla_\pi h_i(u,\pi)+w_i-\sum_{\tau\in I_0}y_\tau\nabla_\pi k_\tau(u,\pi)\right)\right)<\sum_{\alpha=1}^{\mu}\beta_\alpha d(x,u), \qquad (3.15)$$

which contradicts (3.9). Hence, (3.6) and (3.7) cannot hold.
If case (b) is satisfied, then we note that (3.12) holds and that (3.11) implies

$$F\left(x,u,\nabla_\pi h_j(u,\pi)+w_j-\sum_{\tau\in I_0}y_\tau\nabla_\pi k_\tau(u,\pi)\right)<-\rho_j d(x,u), \quad \text{for some } j\in P. \qquad (3.16)$$

Since (3.16) and (3.12) imply (3.15), it is clear that (3.6) and (3.7) cannot hold.

Now suppose that case (c) is satisfied. Since $\lambda\in\Lambda^+$, it follows from (3.10) and (3.11) that

$$\sum_{i=1}^{p}\lambda_i\left(f_i(x)+x^{\mathrm{T}}w_i\right)<\sum_{i=1}^{p}\lambda_i\{f_i(u)+u^{\mathrm{T}}w_i+h_i(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi h_i(u,\pi)-$$
$$-\sum_{\tau\in I_0}\left[y_\tau g_\tau(u)+y_\tau k_\tau(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi y_\tau k_\tau(u,\pi)\right]\}.$$

Thus, by (c),

$$F\left(x,u,\sum_{i=1}^{p}\lambda_i\left(\nabla_\pi h_i(u,\pi)+w_i-\sum_{\tau\in I_0}y_\tau\nabla_u g_\tau(u)\right)\right)<-\rho d(x,u),$$

and using the fact that $\sum_{\alpha=1}^{\mu}\beta_\alpha+\rho\geq 0$, we obtain

$$F\left(x,u,\sum_{i=1}^{p}\lambda_i\left(\nabla_\pi h_i(u,\pi)+w_i-\sum_{\tau\in I_0}y_\tau\nabla_u g_\tau(u)\right)\right)<\sum_{\alpha=1}^{\mu}\beta_\alpha d(x,u),$$

which contradicts (3.9). Hence, (3.6) and (3.7) cannot hold.

A Special Case

Let us consider the compact convex sets C_i to be defined by

$$C_i=\left\{B_i w \mid w^{\mathrm{T}}B_i w\leq 1\right\}.$$

It is easily shown that $\sqrt{x^{\mathrm{T}}B_i x}=s(x\mid C_i)$. In this case, the problems (NMP) and (NMD) can be rewritten as follows:

$$\text{minimize: } \left\{f_1(x)+\sqrt{x^{\mathrm{T}}B_1 x},\ldots,f_p(x)+\sqrt{x^{\mathrm{T}}B_p x}\right\}$$
$$\text{subject to: } g(x)\geq 0,\ x\in D. \qquad \text{(NMP*)}$$

and

$$\left.\begin{aligned}\text{maximize:}\quad \{ & f_1(u)+h_1(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi h_1(u,\pi)+u^{\mathrm{T}}B_1 w-\\ & -\sum_{i\in I_0}\left[y_i g_i(u)+y_i k_i(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi y_i k_i(u,\pi)\right],\\ & \ldots\\ & f_p(u)+h_p(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi h_p(u,\pi)+u^{\mathrm{T}}B_p w-\\ & -\sum_{i\in I_0}\left[y_i g_i(u)+y_i k_i(u,\pi)-\pi^{\mathrm{T}}\nabla_\pi y_i k_i(u,\pi)\right]\},\end{aligned}\right\} \qquad \text{(NMD*)}$$

subject to

$$\lambda^{\mathrm{T}}\nabla_{\pi}h(u,\pi)+\sum_{i=1}^{p}\lambda_i B_i w=\nabla_{\pi}\left(y^{\mathrm{T}}k(u,\pi)\right),$$

$$\sum_{i\in I_\alpha}\left[y_i g_i(u)+y_i k_i(u,\pi)-\pi^{\mathrm{T}}\nabla_{\pi}\left(y_i k_i(u,\pi)\right)\right]\le 0,\quad \alpha=1,\dots,\mu,$$

$$y\ge 0,\quad w^{\mathrm{T}}B_i w\le 1,\ i\in P,\quad \lambda\in\Lambda^{+}.$$

Remarks

1) If $p=1$ the problems (NMP*) and (NMD*) become (NDP) and (NDHGD) considered by Mishra et. al. in [16].
2) If $p=1$, $I_0=M$ and $I_\alpha=\varnothing$ for $\alpha=1,\dots,\mu$, then (NMP*) and (NMD*) become respectively the problems (NDP) and (NDHMD) considered in [16].
3) If $p=1$, $I_0=\varnothing$, $I_1=M$, and $I_\alpha=\varnothing$ for $\alpha=2,\dots,\mu$, then (NMP*) and (NMD*) become respectively (NDP) and (NDHD) considered in [16].
4) If $h(u,\pi)=\pi^{\mathrm{T}}\nabla f(u)$ and $k_i(u,\pi)=\pi^{\mathrm{T}}\nabla g_i(u),\ i\in M$, then (NMD*) reduces to (VD) considered by Yang, Teo and Yang in [25].
5) When $F\left(x,u,\nabla\varphi(u)\right)=\nabla\varphi(u)^{\mathrm{T}}\eta(x,u)$, where $\eta\ :\ \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ is a vector valued function, then the conditions (a), (b) and (c) of Theorem 3.1 reduce to higher-order generalized invexity considered in [16].

REFERENCES

[1] M.S. Bazaraa, J.J. Goode, *Oper. Research* **21** (1973), 1-9.
[2] C.R. Bector, B.K. Bector, Generalized B-invex functions and second order duality for a non-linear programming problem. *Congressus Numeratium* **22** (1986), 37-52.
[3] C.R. Bector, S. Chandra, Second order symmetric and self dual programs. *Opsearch* **23** (1986), 89-95.
[4] S. Chandra, Abha, *Europ. J. Oper. Res.* **122** (2000), 161-165.
[5] X. Chen, Higher-order symmetric duality in nondifferentiable multiobjective programming problems. *J. Math. Anal. Appl.* **290** (2004), 423-435.
[6] B.D. Craven, Lagrangian conditions and quasiduality. *Bull. Austral. Math. Soc.* **16** (1977), 325-339.
[7] G. Devi, Symmetric duality for nonlinear programming problem involving η-bonvex functions. *European J. Oper. Res.* **104** (1998), 615-621.
[8] M.A. Hanson, *Opsearch* **30** (1993), 311-320.
[9] M.A. Hanson, B. Mond, *J. Inf. Optimiz. Sci.* **3** (1982), 25-32.
[10] S.H. Hou, X.M. Yang, On second-order symmetric duality in nondifferentiable programming. *J. Math. Anal. Appl.* **255** (2001), 488-491.
[11] V. Jeyakumar, *Utilitas Mathematics* **34** (1985), 1-15.
[12] D.J. Mahajan, Contributions to optimality conditions and duality theory in nonlinear programming. Ph.D. Thesis, 1977.
[13] O.L. Mangasarian, Second and higher order duality in nonlinear programming. *J. Math. Anal. Appl.* **51** (1975), 607-620.
[14] S.K. Mishra, *Europ. J. Oper. Res.* **127** (2000), 507-518.
[15] S.K. Mishra, N.G. Rueda, Higher order generalized invexity and duality in mathematical programming. *J. Math. Anal. Appl.* **247** (2000), 173-182.
[16] S.K. Mishra, N.G. Rueda, Higher order generalized invexity and duality in nondifferentiable mathematical programming problems. *J. Math. Anal. Appl.* **272** (2002), 496-506.
[17] B. Mond, Second-order duality for nonlinear programming. *Opsearch* **11** (1974), 90-99.
[18] B. Mond, T. Weir, Generalized convexity and duality. In: S. Schaible, W.T. Ziemba (Eds.), Generalized Convexity in Optimization and Economics, Academic Press, New York, 1981, pp. 263-280.
[19] B. Mond, J. Zhang, Higher order invexity and duality in mathematical programming. In: J.P. Crouzeix, et al. (Eds.), Generalized Convexity, Generalized Monotonicity: Recent Results, Kluwer Academic, Dordrecht, 1998, pp. 357-372.
[20] S. Nanda, L.N. Das, *Europ. J. Oper. Res.* **88** (1996), 572-577.
[21] S. Pandey, Duality for multiobjective fractional programming involving generalized η-bonvex functions. *Opsearch* **28** (1991), 36-43.
[22] V. Preda, Duality for multiobjective fractional programming problems involving n-set functions. In: C. Andreian Cazacu, W.E. Lehto and T.M. Rassias (Eds.), Analysis and Topology, 1998, pp. 569-583.
[23] R.T. Rockafellar, Convex Analysis. Princeton Univ. Press, 1970.
[24] X.Q. Yang, *Math. Prog.* **81** (1998), 327-347.
[25] X. M. Yang, K. L. Teo, X. Q. Yang, Duality for a class of nondifferentiable multiobjective programming programs. *J. Math. Anal. Appl.* **252** (2000), 999-1005.
[26] X.M. Yang, K.L. Teo, X.Q. Yang, Higher order generalized convexity and duality in nondifferentiable multiobjective mathematical programming. *J. Math. Anal. Appl.* **297** (2004), 48-55.
[27] X. M. Yang, X.Q. Yang, K. L. Teo, S. H. Hou, Second order duality for nonlinear programming. *Indian J. Pure Appl. Math.* **35**(5) (2004), 699-708.
[28] J. Zhang, B. Mond, Second order B-invexity and duality in mathematical programming. *Utilitas Mathematica* **50** (1996), 19-31.