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DYNAMIC LYAPUNOV INDICATOR: A PRACTICAL TOOL FOR DISTINGUISHING BETWEEN ORDERED AND CHAOTIC ORBITS IN CONTINUOUS DYNAMICAL SYSTEMS

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***Abstract**: In the present work our goal was to verify if the Dynamic Lyapunov Indicator (DLI), proposed recently by Saha and Budhraja as a new tool for distinguishing between ordered and chaotic orbits, gives correct conclusions when is applied to continuous dynamical systems. The behavior of certain continuous dynamical systems, like Ueda oscillator, Rossler oscillator, Rucklidge oscillator and Thomas oscillator has been studied and conclusions regarded DLI for ordered/chaotic orbits has been considered. The simplicity of the idea and the correlation between the conclusions obtained by DLI and other tools, show that DLI is a very consistent indicator in identifying ordered/chaotic orbits in continuous dynamical systems.*
***Keywords**: indicator of chaos, continuous dynamical systems*

1. INTRODUCTION

Although here is no universality accepted definition of chaos, most experts would concur that chaos is the aperiodic, long-term behavior of a bounded, deterministic system that exhibits sensitive dependence on initial conditions. Deterministic nonlinear systems which exhibit this ubiquitous phenomenon are said to be chaotic system. Common chaotic systems are categorized into two groups: maps and flows.

Chaos theory is a scientific discipline that focuses on the study of nonlinear dynamical systems that are highly sensitive to initial conditions. Today, chaos theory is applied in many other scientific disciplines: mathematics, biology, computer science, economics, engineering, finance, philosophy, physics, politics, population dynamics and robotics. Appearance of chaos has been identified through various indicators in the past. We mention here only the well-known tools: the time series method, phase-plane or phase-space method, Lyapunov exponents, bifurcation diagram and the Poincare section of surface.

Some recent tools seem to be more efficient for distinguishing between chaotic or regular orbits, especially in higher dynamical systems. First Lyapunov Indicator (FLI) was introduced by Froeschle et al. [1] and applied to the structure of a steroidal belt. Soha et al. applied the FLI to study certain discrete maps like Tinkerbell map, Ikeda map and Duffing map [2]. Other discrete maps (Gaussian map, Delayed logistic map, 2-D and 4-D Froeschle map) have been studied by Deleanu [3]. Smaller Alignment Indices (SALI) was introduced by Skokos in 2001 [6] and has been successfully applied to some symplectic map and some Hamiltonian flows. Gottwald and Melbourne (2004, 2005) have introduced 0-1 test [7]. Dynamic Lyapunov Indicator (DLI) has been introduced by Saha et al. (2007) and applied for various discrete maps [4, 5]. They mentioned that, before accepting DLI as an indicator of regularity and chaos, other studies are necessary, especially for continuous dynamical systems. This is the reason of this paper.

This paper is organized as follows. In Section 2 we recall the definition of FLI and DLI. In Section 3 we compute these indicators for some regular and chaotic orbits of four continuous dynamical systems. Conclusions and future developments are provided in Section 4.

2. INDICATORS OF CHAOS: FLI AND DLI

2.1 Fast Lyapunov Indicator (FLI)

The FLI is defined as follows:

Starting with a m-dimensional basis $V_m(0) = \left(v_1(0), v_2(0), ..., v_m(0)\right)$ embedded in an n-dimensional space with an initial condition $\left(x_1(0), x_2(0), ..., x_n(0)\right)$ we take at each iteration the largest amongst the vectors of the evolving basis. Thus, the FLI is defined as:

$$FLI = \sup\left\|v_j\right\|,\ j = 1,2,..,m \qquad (1)$$

Froeschle has shown that FLI increases exponentially for chaotic orbits and decreases to zero or present a linear variation for a regular orbit.

2.2 Dynamic Lyapunov Indicator (DLI)

Saha and Budhraja are defined DLI as follows:

Let J be the Jacobian matrix of a dynamical system. For some discrete time, we calculate the eigenvalues of the matrix J and then plot the largest eigenvalue:

$$\lambda_{\max} = \max\left|\lambda_j\right|,\ j = 1,2,...,n \qquad (2)$$

where λ_j are the solution of $\left|J - \lambda \cdot I_n\right| = 0$. It seems that these eigenvalues form a definite pattern for regular motion and are distributed randomly for chaotic orbits.

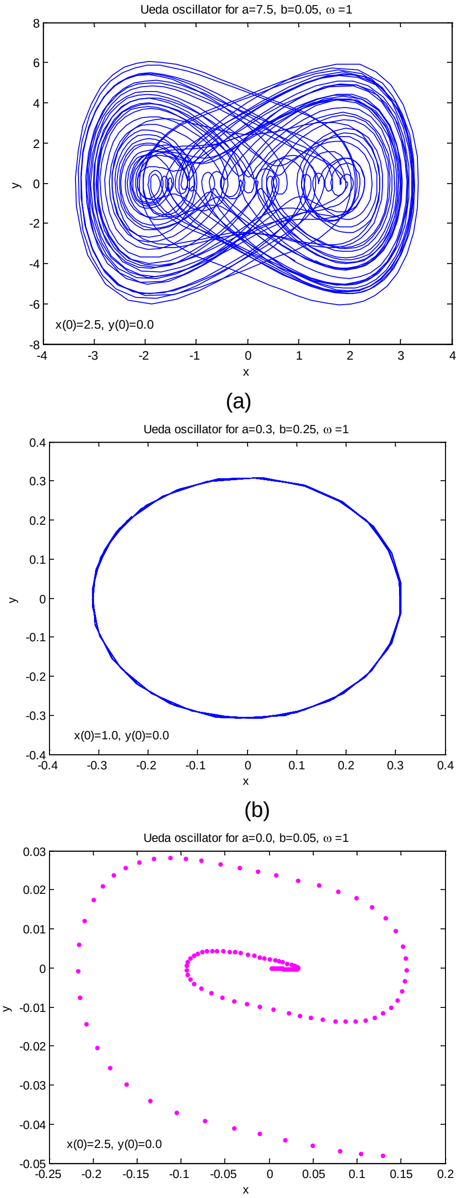


Figure 1: Phase-plane x-y for Ueda oscillator
a) $a = 7.5, b = 0.05, \omega = 1$ **; b)** $a = 0.3, b = 0.25, \omega = 1$**; c)** $a = 0.0, b = 0.05, \omega = 1$

3. NUMERICAL RESULTS

We have applied above defined indicators for the models given below:

3.1. Ueda oscillator

Chaotic oscillations of periodically driven nonlinear oscillator were studied in some details around 1980 by the Japanese researcher Ueda. Systems of such kind can be realized as mechanical and electronic devices. The equation of motion can be transformed into the first order system of differential equations:

$$\frac{dx}{dt} = y \quad , \quad \frac{dy}{dt} = -x^3 - b\,y + a\,\sin\omega t \qquad (3)$$

with a, b and ω real constants.

For $a = 7.5, b = 0.05, \omega = 1$ and initial conditions $(x_0, y_0) = (2.5, 0.0)$ the trajectory corresponds to a chaotic motion (see Fig. 1a). Indeed, the exponential increase of FLI in Fig. 2a indicates that the orbit is chaotic. The ordinate is taken with base 10. A random distribution for the DLI is obvious from Fig. 3a.Changing the parameter values to $a = 0.3, b = 0.25, \omega = 1$ and keeping the initial conditions the trajectory evolves to a limit cycle (Fig. 1b). FLI tends to zero (Fig. 2b) and a pattern in DLI is clearly visible for this case (Fig. 3b).

The trajectory evolves to the fixed point (0, 0) for $a = 0.0, b = 0.05, \omega = 1$ (Fig. 1c). At the beginning, in the transition period $t \in [0, 300]$, FLI increases to about 10^{250} but then it decreases quickly to zero (fig. 2c). The transition period is necessary for DLI to reach the constant value $DLI = 0.05$ (Fig. 3c).

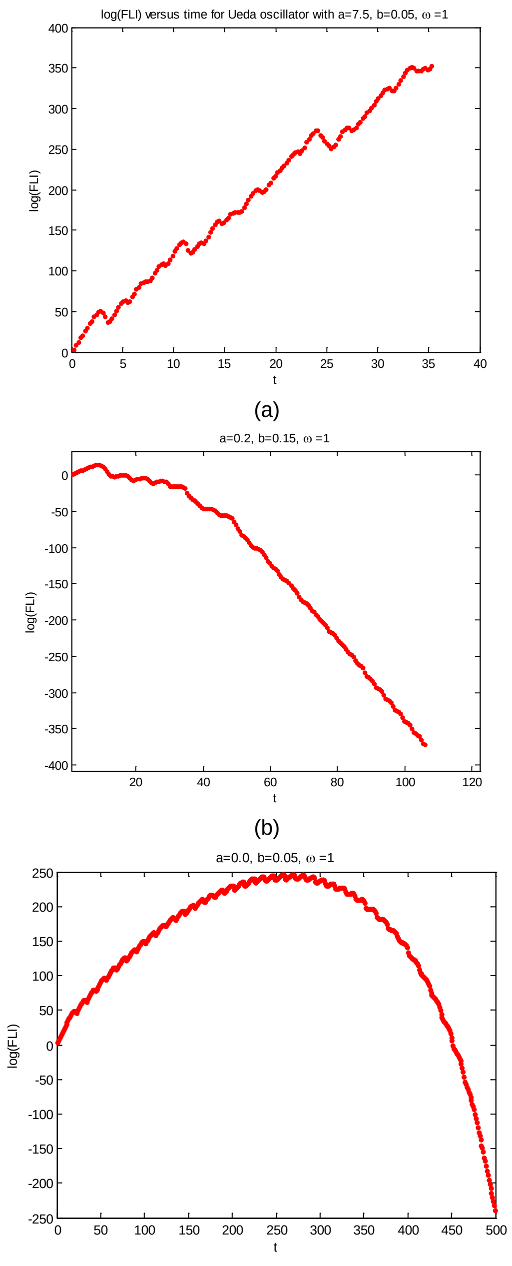


Figure 2: Log(FLI) plots for Ueda oscillator
a) $a = 7.5, b = 0.05, \omega = 1$ **; b)** $a = 0.3, b = 0.25, \omega = 1$**; c)** $a = 0.0, b = 0.05, \omega = 1$

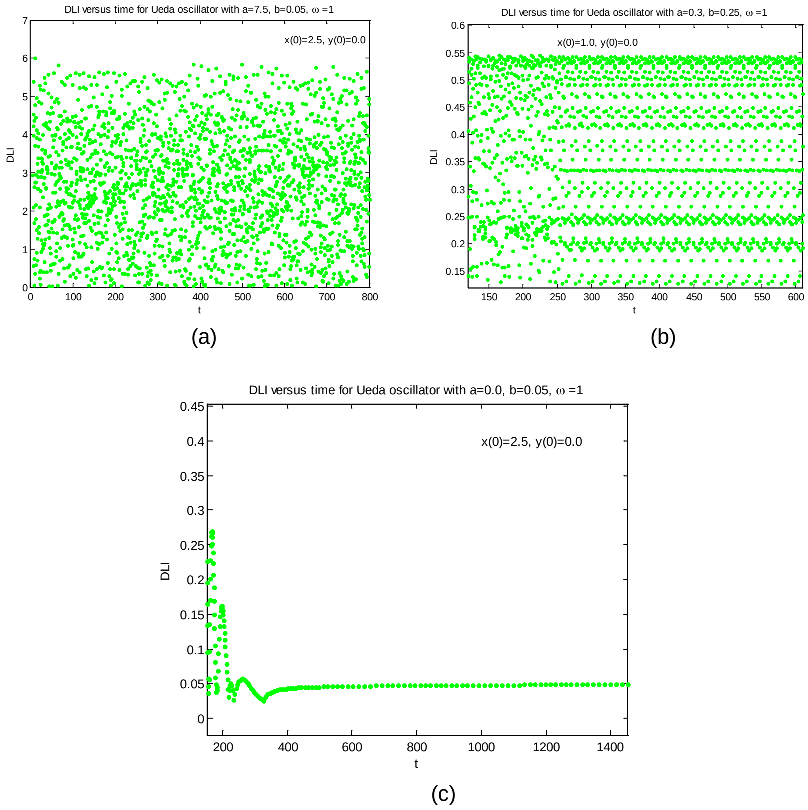


Figure 3: DLI plots for Ueda oscillator
a) $a = 7.5, b = 0.05, \omega = 1$; **b)** $a = 0.3, b = 0.25, \omega = 1$; **c)** $a = 0.0, b = 0.05, \omega = 1$

3.2 Rossler oscillator

The so called "Rossler" system is credited to Otto Rossler and arose from work in chemical kinetics. The system is described with three coupled non-linear differential equations:

$$\frac{dx}{dt} = -y - z, \quad \frac{dy}{dt} = x + ay, \quad \frac{dz}{dt} = b + z(x - c) \qquad (4)$$

The main features of the system can be deduced with non-linear methods such as Poincare maps and bifurcation diagrams. It is well-known that a bifurcation diagram is created by running the equations of the system, holding all but one of the variables constant and varying the last one. Then, a graph is plotted of the points that a particular value for the changed variable visits, after transient factors have been neutralized. Chaotic regions are indicated by filled-in regions of the plot.

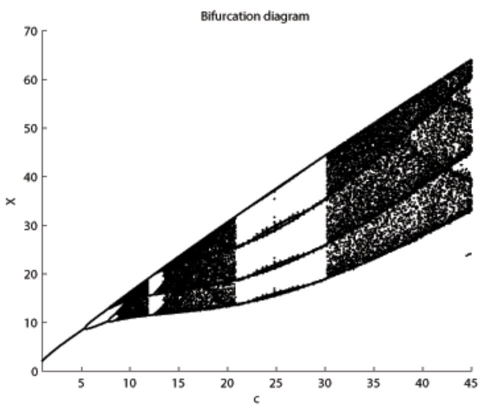


Figure 4: The bifurcation diagram for the Rossler oscillator

A graphical illustration of the changing attractor over a range of values is observed in Fig. 5.

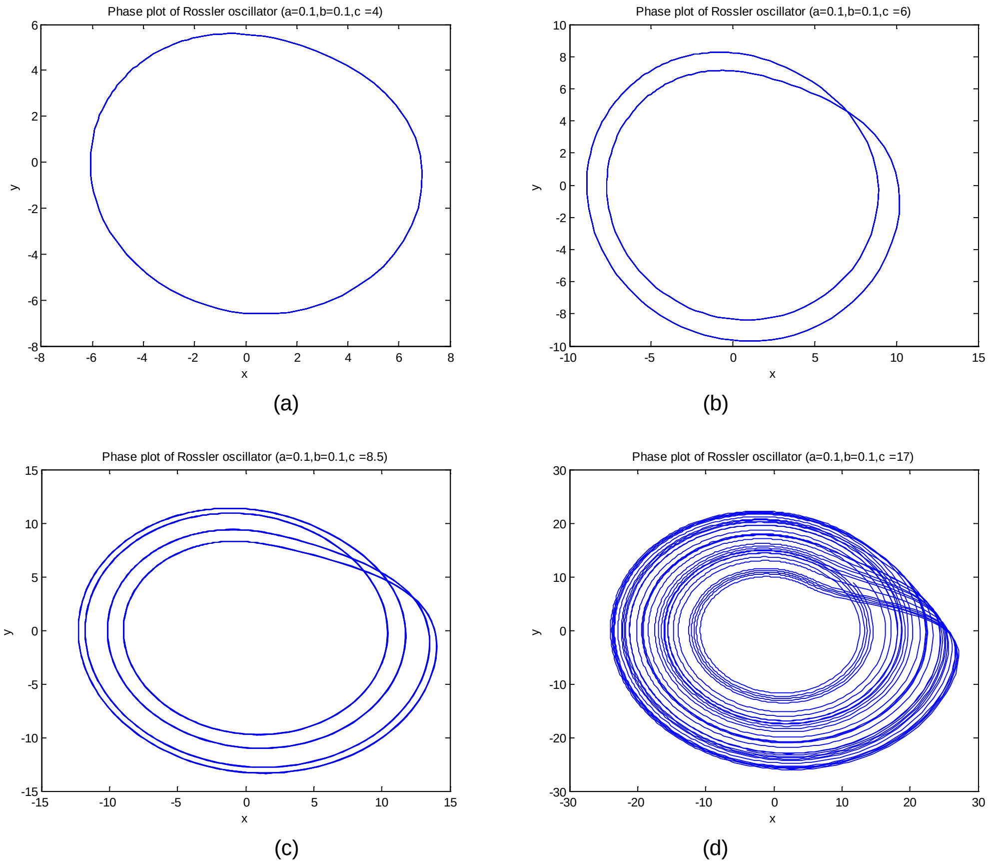
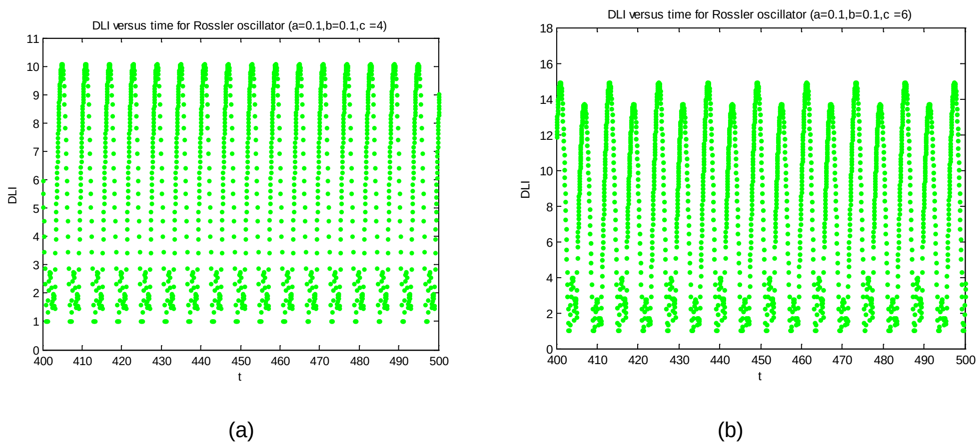
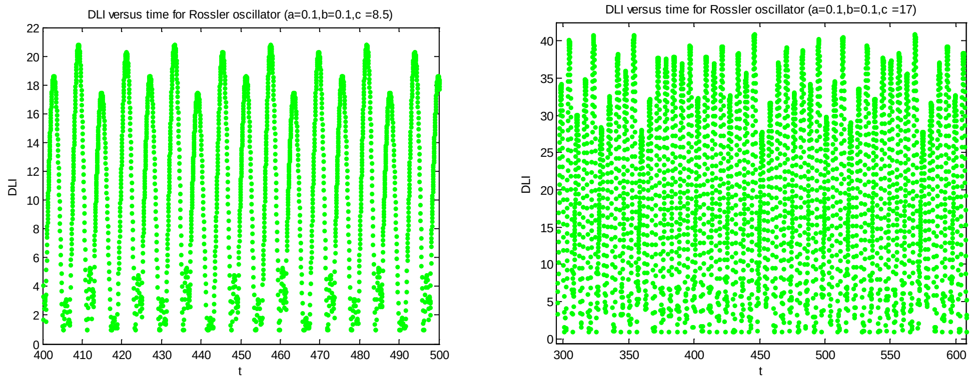


Figure 5: Phase-plane x-y for Rossler oscillator
a) $c = 4$; **b)** $c = 6$; **c)** $c = 8.5$; **d)** $c = 17$

In our study, we chosen $a = b = 0.1$ and *c* was changed. The bifurcation diagram reveals that low values of *c* are periodic, but quickly became chaotic as *c* increases (see Fig. 4). This pattern repeats itself as *c* increases – there are sections of periodicity interspersed with periods of chaos, and the trends is towards higher-period orbits as *c* increases.

As Saha and Budhraja said, the DLI plots form definite patterns in regular cases and are distributed randomly in chaotic case (Fig. 6). It is interesting to observe the relationship between the DLI plots and the period of motion.





(c) (d)

Figure 6: DLI plots for Rossler oscillator
a) $c = 4$; **b)** $c = 6$; **c)** $c = 8.5$; **d)** $c = 17$

In Fig. 7 it is displayed the evolution of log (FLI) for three cases, namely $c = 8.5$ (period-4 orbit), $c = 13$ (sparse chaotic orbit), $c = 17$ (filled-in chaotic orbit). In all plots the FLI increase in time, following however completely different time rates (but if we consider separately these plots, it would be difficult to specify the type of orbit).

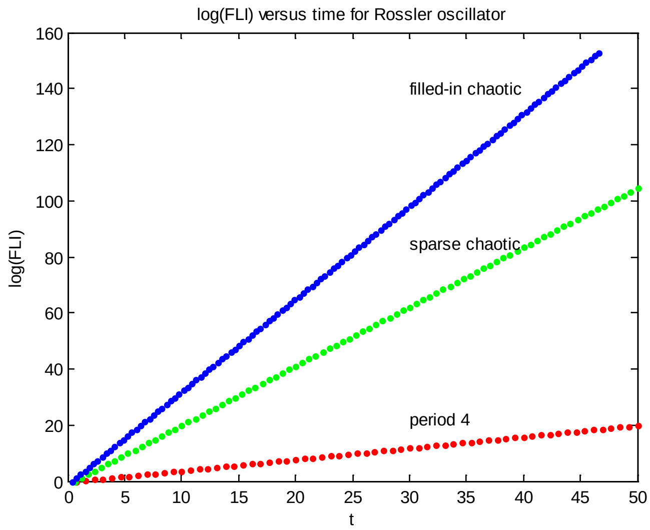


Figure 7: Log(FLI) plots for Rossler oscillator

3.3. Rucklidge oscillator

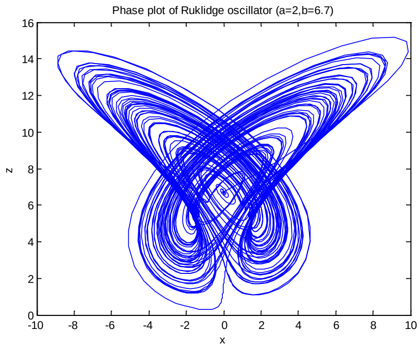
The following nonlinear autonomous ordinary differential equations comprise the Rucklidge system:

$$\frac{dx}{dt} = -ax + by - yz, \quad \frac{dy}{dt} = x, \quad \frac{dz}{dt} = -z + y^2 \qquad (5)$$

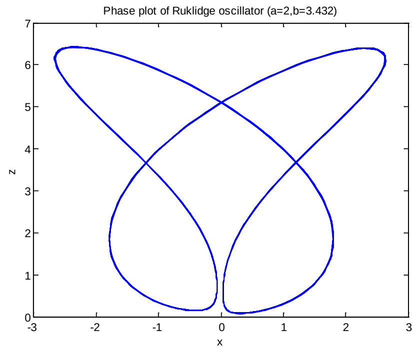
For parameter values $a = 2, b = 6.7$ and initial conditions $(x_0, y_0, z_0) = (1.0, 0.0, 4.5)$ we get a chaotic orbit. This can be observed through the phase-plane x-z in Fig. 8a. As expected, DLI plots present a random distribution (Fig. 8b).

Changing *b* from 6.7 to 3.432 and keeping other parameters same, the system displays regular behavior (more precisely a limit cycle), as we can see in Fig. 8b. A pattern is clearly visible in DLI plots (Fig. 9b). In the end, the system evolves to a fixed point for $a = 2, b = 1.7$. DLI, after a short transition period, tends to constant value $DLI \cong 2.724$.

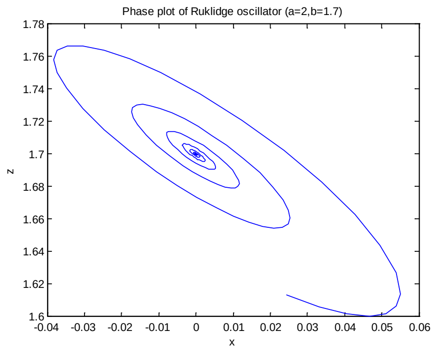
The evolution of log (FLI) with time is similar with that for Rossler oscillator.



(a)



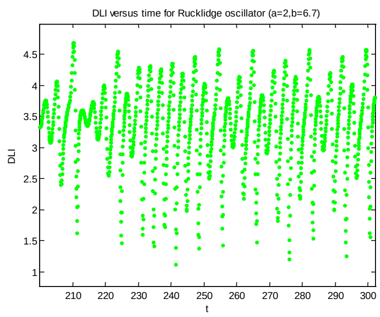
(b)



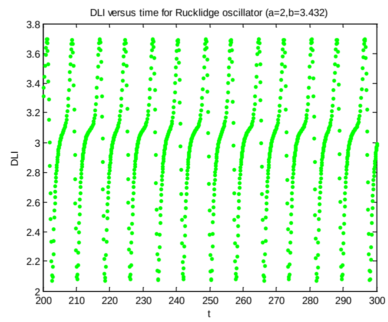
(c)

Figure 8: Phase-plane x-z for Rucklidge oscillator

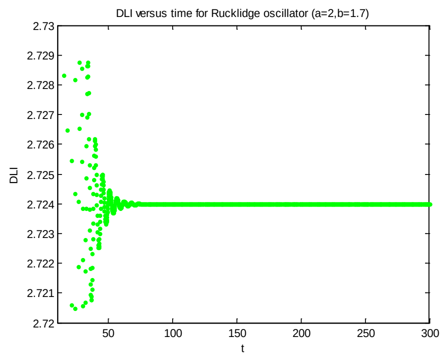
a) $a = 2,\ b = 6.7$; **b)** $a = 2, b = 3.432$; **c)** $a = 2, b = 1.7$



(a)



(b)



(c)

Figure 9: DLI plots for Rucklidge oscillator

a) $a = 2,\ b = 6.7$; **b)** $a = 2, b = 3.432$; **c)** $a = 2, b = 1.7$

3.4 Thomas cyclically symmetric oscillator

As the last example we get the 3-D autonomous Thomas cyclically symmetric oscillator, whose behavior is described by the next set of differential equations:

$$\frac{dx}{dt} = -b\,x + \sin y\,, \quad \frac{dy}{dt} = -b\,y + \sin z\,, \quad \frac{dz}{dt} = -b\,z + \sin x$$

6) where *b* is a real constant. This system evolves chaotically for $b = 0.18$ (see Fig. 10 a, b) and regularly for $b = 0.31$ (see Fig. 11 a, b). DLI plots confirm these evolutions.

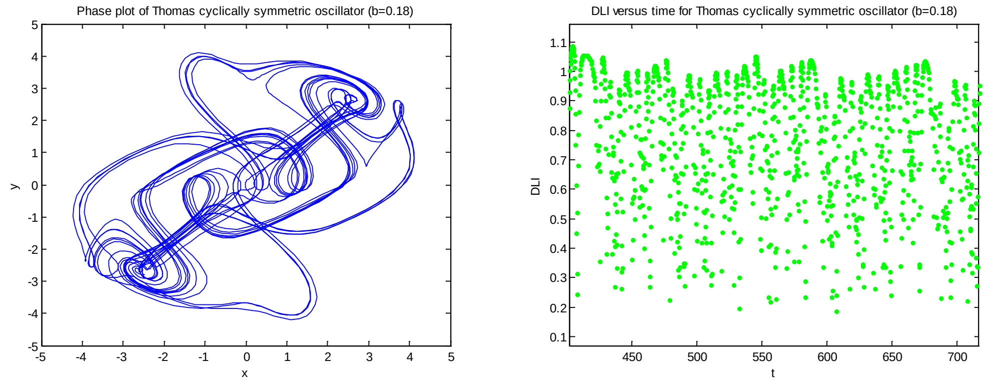


Figure 10: Phase plot x –y and DLI plot for Thomas cyclically symmetric oscillator with $b = 0.18$

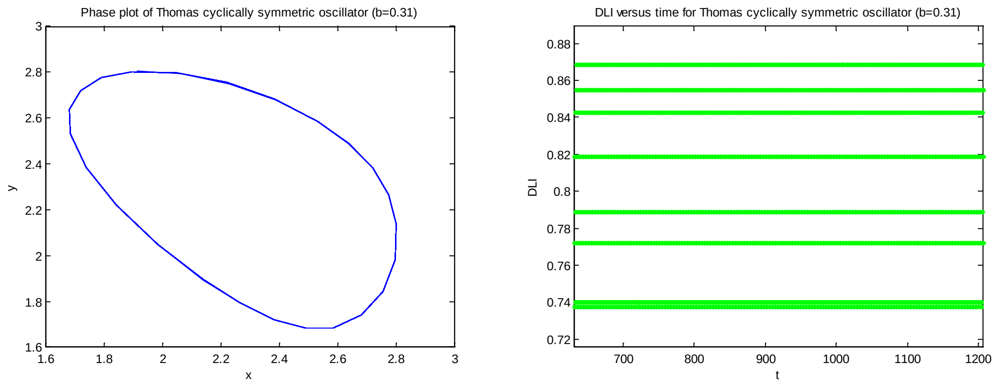


Figure 11: Phase plot x –y and DLI plot for Thomas cyclically symmetric oscillator with $b = 0.31$

4. CONCLUSIONS

The goal of this work was to apply the DLI method for distinguishing between ordered and chaotic orbits in the case of some continuous-time dynamical systems. We investigated the 2-D Ueda oscillator and the 3-D Rossler, Rucklidge and Thomas oscillators. The main conclusions of the study are:

a) The DLI behaves randomly for chaotic orbits and regularly for ordered orbits. In this last case, there are an interesting relationship between the DLI plots and the period of motion;

b) Generally, the FLI increases exponentially for chaotic orbits and decreases to zero or presents a linear variation for a regular orbits;

c) The DLI gives very clear indication about orbits nature whenever applied; Though DLI has significant merits, it does not clearly identify regular and chaotic orbits in some cases;

d) The computation of FLI and DLI is fast and easy; Only few hundreds of iterations are sufficient to get a conclusion;

e) For numerical integrations we have used modern software, i.e. Matlab package, where the possibility of occurring of round off error be minimum;

f) Before accepting DLI as an indicator of chaos, we think that other studies are necessary, especially for limit cases

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