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Incompressible models and low mach number expansions

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Abstract. In the field of CFD, the incompressibility assumption is very important for applications since many common fluids (liquids) are incompressible or only very slightly compressible. The purpose of this paper is to present some mathematical properties of incompressible models aim and the way of expressing this type of number expansions.

1. Introduction

Mathematically, the incompressibility condition means:

$$divV = 0 \qquad (1.1)$$

Therefore, the volume occupied by a group of fluid particles at the initial time remains constant during the flow. The continuity equation written as:

$$\rho_t + u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z) = 0$$

leads to

$$\frac{D\rho}{Dt} = \rho_t + V \cdot \nabla\rho = 0 \qquad (1.2)$$

This means that if the density is constant initially and on the boundaries from where the fluid comes inside the domain under consideration it remains so. This is equivalent to say that the fluid is homogeneous.

Further, having in mind the previous derivation of the compressible models and their EOS and constitutive models, it could be useful to make a distinction between models obtained by:

- Incompressibility hypothesis.
- Low Mach number expansions.

We emphasize here that the incompressibility hypothesis does not impose an explicit restriction on the magnitude of the velocity. Moreover, the incompressible models aim to describe liquids where compression effects are neglected and the density is taken as constant. For example, it is not rational to expect from an incompressible model to describe accurately the propagation of acoustic or pressure waves through liquids. This is due the fact that the incompressibility condition (1.1) is normally associated with other working hypothesis made on the EOS and on the fluid transport properties. Further, the energy equation is firmly 'decoupled' from the continuity and momentum equations, by stating that the temperature field can be calculated separately, after the velocity and pressure fields have been determined.

On the other side, the compressible models presented in the previous chapter are valid for gases. Starting from the compressible models it is rational to discuss about the low Mach number expansions. The precise definition of the Mach number is: $M = \frac{|V|}{\sqrt{p'(\rho)}}$. Therefore, letting M go to zero means that, keeping constant values for the density and the temperature, the magnitude of the velocity is a small parameter. An asymptotic analysis starting from the Navier-Stokes equation derived for a compressible ideal gas shows that there is a lack of consistency between compressible models and 'incompressible' submodels, in the presence of heat conduction. A possible physical explanation is the following assertion: compressible models are valid for gases and the low Mach number limit yields particular incompressible submodels. These particular submodels are definitively determined by the EOS and transport properties chosen for the gas. Further, such an asymptotic analysis reveals that the incompressible submodel is very sensitive to the errors in the pressure calculation.

In what follows, we assume the fluid to be incompressible, homogeneous, non-heat conducting and viscous, with constant coefficient of viscosity η. Body forces are also neglected, only for simplicity. We study three mathematical formulations of the governing equations in Cartesian coordinates and restrict our attention to the two-dimensional case.

1.2 The Incompressible Navier-Stokes Equations in Primitive Variable Form

The primitive variable formulation of the incompressible two-dimensional Navier-Stokes equations is given by:

$$\begin{aligned} u_x + v_y &= 0 \\ u_t + uu_x + vu_y + \frac{1}{\rho}p_x &= v\left[u_{xx} + u_{yy}\right] \\ v_t + uv_x + vv_y + \frac{1}{\rho}p_y &= v\left[v_{xx} + v_{yy}\right] \end{aligned} \tag{1.3}$$

where the kinematic viscosity is:

$$v = \frac{\eta}{\rho} \tag{1.4}$$

Recall that η is the coefficient of shear viscosity. We have a set of three equations for the three unknowns *u, v, p,* the primitive variables. This is a mixed elliptic-parabolic system. Due to the mixed nature of the mathematical model, the solution cannot be obtained directly via time-marching algorithms. In principal, given a domain along with initial and boundary conditions for the equations one should be able to solve this problem using the primitive variable formulation.

1.3 The Incompressible Navier-Stokes Equations in Stream-Function Vorticity Form

The stream-function vorticity formulation is another way of expressing the incompressible Navier-Stokes equations. This formulation is attractive for the two-dimensional case but not so much in three dimensions, in which the role of a stream function is replaced by that of a vector potential. The magnitude of the vorticity vector can be written as:

$$\varsigma = v_x - u_y \tag{1.5}$$

Introducing a stream function Ψ we have for the velocity components: $u = \psi_y, v = -\psi_x$. By combining the momentum equations so as to eliminate the pressure p, and using **Error! Reference source not found.** we obtain the vorticity transport equation:

$$\varsigma_t + u\varsigma_x + v\varsigma_y = v[\varsigma_{xx} + \varsigma_{yy}] \qquad (1.6)$$

This is an advection-diffusion equation of parabolic type. In order to solve it, one requires the solution for the stream function ψ, which is in turn related to the vorticity ς via:

$$\psi_{xx} + \psi_{yy} = -\varsigma \qquad (1.7)$$

This is called the Poisson equation and is of elliptic type. There are numerical schemes to solve (1.5-1.7) using the apparent decoupling of the parabolic-elliptic problem to transform it into the parabolic equation or the vorticity and the elliptic equation for the stream function. A relevant observation, from the numerical point of view, is that the convection terms of the left-hand side of equation (1.6) can be written in conservative form and hence we have:

$$\varsigma_t + (u\varsigma)_x + (v\varsigma)_y = v[\varsigma_{xx} + \varsigma_{yy}] \qquad (1.8)$$

This follows from the fact that $u_x + v_y = 0$, which was also used to obtain.

1.3 The Incompressible Navier-Stokes Equations in Artificial Compressibility Form

The artificial compressibility formulation is yet another approach to formulate the incompressible Navier-Stokes equations and was originally put forward by Chorin, for the steady case. Let us consider the two-dimensional incompressible Navier-Stokes equations written in non-dimensional form:

$$u_x + v_y = 0 \qquad (1.9)$$

$$u_t + uu_x + vu_y + p_x = \alpha[u_{xx} + u_{yy}]$$
$$v_t + uv_x + vv_y + p_y = \alpha[v_{xx} + v_{yy}] \qquad (1.10)$$

where the following non-dimensionalisation has been used:

$$u \leftarrow \frac{u}{V_\infty}, v \leftarrow \frac{v}{V_\infty}, p \leftarrow \frac{p}{\rho_\infty V_\infty^2},$$
$$x \leftarrow \frac{x}{L}, y \leftarrow \frac{y}{L}, t \leftarrow \frac{tV_\infty}{L},$$
$$\alpha = \frac{1}{R_{eL}}, R_{eL} = \frac{V_\infty L}{V_\infty}.$$

Multiplying by the non-zero parameter $c^2$and adding an artificial compressibility term $p_t$the first equations reads:

$$p_t + (uc^2)_x + (vc^2)_y = 0 \qquad (1.11)$$

By using equation (1.11) the convective terms in the momentum equations can be written in conservative form, so that the modified system becomes:

$$p_t + (uc^2)_x + (vc^2)_y = 0$$
$$u_t + (u^2 + p)_x + (uv)_y = \alpha\left[u_{xx} + u_{yy}\right]$$
$$v_t + (uv)_x + (v^2 + p)_x = \alpha\left[v_{xx} + v_{yy}\right] \qquad (1.12)$$

The equations can be written in compact form as

$$U_t + F_x(U) + G_y(U) = S(U) \qquad (1.13)$$

where the vectors of unknowns, fluxes and source terms are:

$$U = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, F = \begin{bmatrix} c^2 u \\ u^2 + p \\ uv \end{bmatrix}, G = \begin{bmatrix} c^2 v \\ uv \\ v^2 + p \end{bmatrix}, S = \begin{bmatrix} 0 \\ \alpha(u_{xx} + u_{yy}) \\ \alpha(v_{xx} + v_{yy}) \end{bmatrix} \qquad (1.14)$$

The above equations are called the artificial compressibility equations. Here c^2 is the artificial compressibility factor, usually taken as a constant parameter. The ‘source’ term vector in this case is a function of second derivatives. Note that the modified equations are equivalent to the original equations in the steady state limit only. The left-hand side of the artificial compressibility equations form a non-linear hyperbolic system.
More recently, new formulations have been proposed for the solution of steady and unsteady incompressible Navier-Stokes equations. Since time-marching methods cannot be applied directly, the system (1.12) must be transformed into a more convenient one. The dual time approach requires the addition of derivatives of a fictitious pseudo-time τ to each of the three equations to give:

$$\frac{1}{\beta^2} p_\tau + p_t + (u)_x + (v)_y = 0$$
$$u_\tau + u_t + (u^2 + p)_x + (uv)_y = \alpha\left[u_{xx} + u_{yy}\right]$$
$$v_\tau + v_t + (uv)_x + (v^2 + p)_x = \alpha\left[v_{xx} + v_{yy}\right] \qquad (1.15)$$

where β is a parameter and the term added to the continuity equation has the same form as the basic artificial compressibility method. A steady-state solution in pseudo-time $\left(\frac{\partial p}{\partial \tau}, \frac{\partial u}{\partial \tau}, \frac{\partial v}{\partial \tau} \to 0\right)$ corresponds to an instantaneous unsteady solution in real time. A recommended value for the parameter β in the case the governing equations are written in dimensionless form is $\beta \sim \Theta(1)$. The convective part of the system (1.15) is of hyperbolic type and therefore a time-marching solution procedure is possible.

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