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On a method of estimating chaos control parameters from time series

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Abstract. The algorithm of Ott, Grebogi and Yorke (OGY) is recognized for its efficiency in controlling chaotic dynamical systems, even if the system's equations are not known and the input data are provided by measured time series in experimental settings. Recently, Santos and Graves (SG) proposed a simple method for estimating the chaos control parameters required by OGY algorithm and applied it to the logistic map. Using only two time series of 100 values, they obtained approximate results for the fixed point case within 2 % of the analytical ones. Although the outputs refer only to a particular case, their conclusion seems to be that the method works as well as in general. To check this statement, we performed a large amount of numerical simulations on different one – dimensional maps. With slight different nuances, our findings were the same so we only presented in the paper the logistic map case. We have noticed that the use of only two short time series implies high risks in a reasonable estimate of the location of the fixed points and of the two control parameters (especially of the second). For large enough number of time series (three or five sets of 400 values each, in the paper) the results provided by numerical simulation approximated the theoretical ones within the limit of a few percent at most. The role played by each method parameter, as the radius for a close encounter of the fixed point or the number of the series and their lengths, is also investigated.

1. Introduction

A large number of physical and biological systems are well described by nonlinear discrete or continuous equations. Commonly, the behavior of these systems depends on one or more parameters having a well – defined significance (e.g. birth or death rates, temperature or pressure, etc.). As a consequence of mutations in the environment, the parameters can change radically, forcing the system to behave very differently from what is desired. Even for the systems modelled by one – dimensional difference equations, as for example logistic map, the spectrum of possible dynamical behaviors is large enough to include fixed points, limit cycle or chaos [1 – 4].

In order to maintain a desired activity (e.g. a fixed point) a number of control mechanisms have been proposed, especially if the system is chaotic. The control mechanism of Ott, Grebogi and Yorke (OGY for short) is one of the most known and used [5, 6]. Its main idea is as follows: a chaotic attractor possesses a large number of unstable periodic orbits (UPOs) embedded within itself. By making small adjustments to an accessible system parameter when the system evolves in a small neighborhood of an UPO, this can be stabilized.

Recently, Santos and Graves propose a simple algorithm to estimate the parameters required by OGY technique, based on a time series analysis [7]. They used only two sets of 100 iterates of the logistic map in order to approximate the control parameters for stabilizing the fixed point within 2% of their analytical values. Their results concerned only one particular case, which of course is insufficient to appreciate the effectiveness of the method.

This paper aims to analyze more thoroughly the level of accuracy of the results provided by the above – mentioned algorithm.

2. Short description of the method

Consider a system whose dynamics is governed by the one – dimensional chaotic map

$$x_{n+1}= f\left(x_n, r\right), x_n \in R, n \geq 1 \tag{1}$$

where $r \in R$ is a parameter, and let x^* be an unstable fixed point of map (1) for the control parameter value r_0. When the system evolves in the proximity of the fixed point, its dynamics is well approximated by the linearized map

$$x_{n+1} \cong x^* + \alpha\left(x_n - x^*\right) + \beta\left(r_n - r_0\right) \tag{2}$$

where

$$\alpha = \left.\frac{\partial f(x,r)}{\partial x}\right|_{x=x^*, r=r_0}, \ \beta = \left.\frac{\partial f(x,r)}{\partial r}\right|_{x=x^*, r=r_0} \tag{3}$$

By fixing the control parameter to the nominal value r_0, one has

$$x_{n+1} - x_n \cong (\alpha - 1)\left(x_n - x^*\right) \overset{not}{=} \Delta_n \tag{4}$$

If the trajectory starting from x_0 is close to the fixed point, then the difference Δ_n between two successive iterates is small (due to slow dynamics of the map around x^*). One collects all the pairs $\left(x_n, x_{n+1}\right)$ for which $\Delta_n < \varepsilon$, where $\varepsilon << 1$ is the radius for a close encounter to be detected. Denoting by Δ_m another variation which fulfills condition $\Delta_m < \varepsilon$, one gets

$$\Delta_m - \Delta_n = (\alpha - 1)\left(x_m - x_n\right), \ x_n \Delta_m - x_m \Delta_n = (\alpha - 1)\left(x_m - x_n\right) x^*$$

thus

$$x^* = \frac{x_n \Delta_m - x_m \Delta_n}{\Delta_m - \Delta_n}.$$

In a long enough time series there exist many close encounters, so an average value will be more appropriate for the fixed point's approximation

$$x^* \cong \left\langle \frac{x_n \Delta_m - x_m \Delta_n}{\Delta_m - \Delta_n} \right\rangle \tag{5}$$

with $\langle . \rangle$ denoting an arithmetic mean. The control parameter results from (4)

$$\alpha \cong 1 + \left\langle \frac{\Delta_n}{x_n - x^*} \right\rangle \tag{6}$$

To estimate the other control parameter, β, the control value r_0 is slightly changed to r_1 and the equation (5) is used to get the new fixed point, x_1^*. By doing this for many r_1 it was found that

$$\beta \cong (1-\alpha)\left\langle \frac{x_1^* - x^*}{r_1 - r_0} \right\rangle \tag{7}$$

The *OGY* method assumes a control strategy that satisfy the relation

$$r_n - r_0 = \gamma\left(x_n - x^*\right) \tag{8}$$

where γ is a constant whose value must be selected so the control goal is achieved. From (2) and (8) it results that

$$x_{n+1} - x^* \cong (\alpha + \beta\gamma)\left(x_n - x^*\right) \tag{9}$$

For $\gamma = -\alpha/\beta$ the system is direct toward the fixed point in just an iteration.

3. Numerical simulations

The algorithm described in the previous section was applied to different one-dimensional maps having biological significance. We discuss here the results obtained for the well-known logistic map only but the similar outputs were obtained for the malignant tumor growth map [3].

The logistic map is given by the difference equation

$$x_{n+1} = f(x_n, r) = r\,x_n\,(1 - x_n),\ n \geq 0 \tag{10}$$

where $r \in [0, 4]$ is a real parameter and $x_n \in [0, 1]$ denotes the i^{th} iterate of the map [2]. Figure 1 presents the bifurcation diagram and the Lyapunov exponent for $r \in [2.5, 4]$.

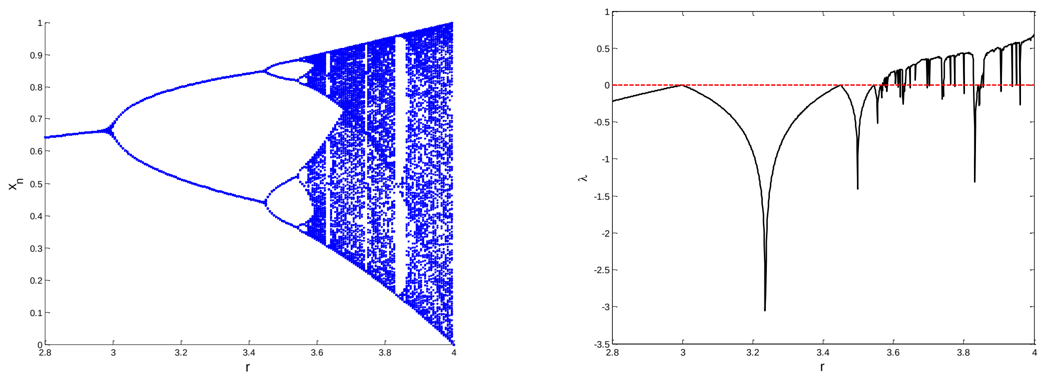

Figure 1. The bifurcation diagram of the logistic map for $r \in [2.5, 4]$ (left) and the corresponding Lyapunov exponent (right)

It results from (10) that $x^* = 1 - \dfrac{1}{r_0}, \alpha = 2 - r_0, \beta = \dfrac{r_0 - 1}{r_0^2}$. Santos and Graves used this map for proving the method's accuracy. By using only two sets of 100 points each (meaning $n = 0, 1, 2, \ldots, 99$) and $r_0 = 3.9, r_1 = 3.91, \varepsilon = 0.1$ (none value for the initial point x_0 was reported) they founded the results presented in Table 1.

At first sight, it seems that this simple method is quite accurate. To check this statement we performed a number of simulations with different parameters' values. Thus, the first 100 iterates of map (10) for $r_0 = 3.7$ and $x_0 = 0.74$ are displayed in Figure 2. Not less than *K* = 25 close encounters of the fixed point $x^* = 0.72973$ were detected.

Table 1: The results reported by Santos and Graves for the logistic map [7]

Parameter	x^*	α	β	γ
Theory	0.74359	- 1.90	0.1907	9.97
Simulation	0.74347	- 1.86	0.1892	9.85
Error	0.016 %	2.1 %	0.79 %	1.2 %

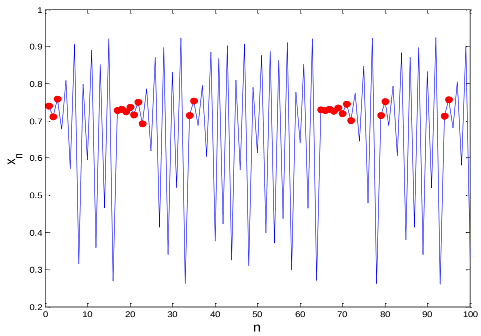

Figure 2. The first 100 iterates of the logistic map with $r_0 = 3.7$ *and* $x_0 = 0.74$. *The red dots stand for the close encounters of the fixed point* $x^* = 0.72973$

This situation changes with the starting point x_0 and with parameter *r*. Thereby, Figure 3 shows the variation of the number *K* of the close encounters of the fixed point with x_0 for $r_0 = 3.7$ and $r_0 = 3.9$. The corresponding Lyapunov exponents are $\lambda_1 = 0.3531$ and $\lambda_2 = 0.4932$. The left panel corresponds to a time series of length *N* = 100 while the right panel presents the same information for *N* = 200. It is useful to note that for $r_0 = 3.9$ (just in the middle of the chaotic area) there are values x_0 for which the trajectories stay away from the fixed point (meaning *K* is zero or very small). For these x_0 and r_0 the method does not work or the errors for parameters x^*, α, β and γ are large.

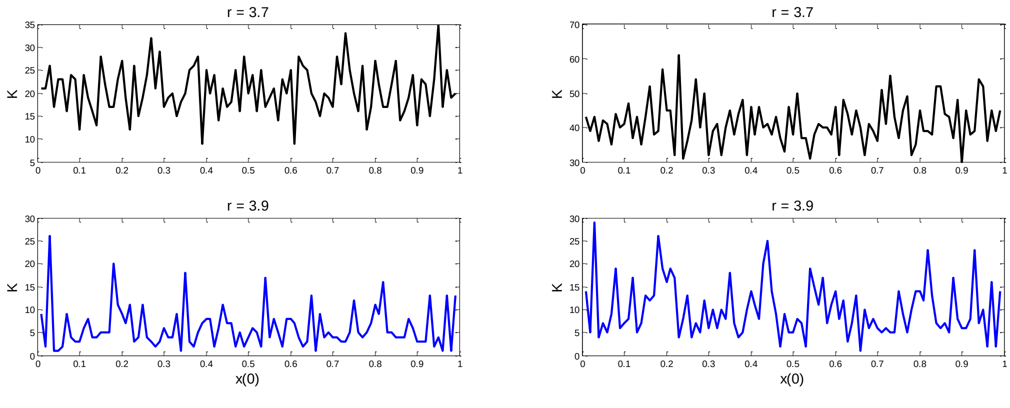

Figure 3. The number K of the close encounters of the fixed point as a function of the starting point x(0) for r = 3.7 and r = 3.9. The time series length was N = 100 (left) or N = 200 (right)

The position of the fixed point x^* can be approximated by equation (5). Figures 4 to 6 reports our findings for $r \in \{3.7, 3.9\}$ and $N \in \{100, 200, 400\}$ as well the relative error $\Delta x^*_{rel} = \left| \frac{x^*_{sim} - x^*_{th}}{x^*_{th}} \right| \cdot 100\%$

For a short time series (N = 100) the errors are less than 0.015 % if r = 3.7, but they increase to 0.18 % for certain initial points in the case r = 3.9. Moreover, the noticeable gaps in the graphical representation show the lack of neighbors for the fixed point. Increasing the time series length the errors in the fixed point location become extremely small and the above mentioned gaps disappear (especially for N = 400).

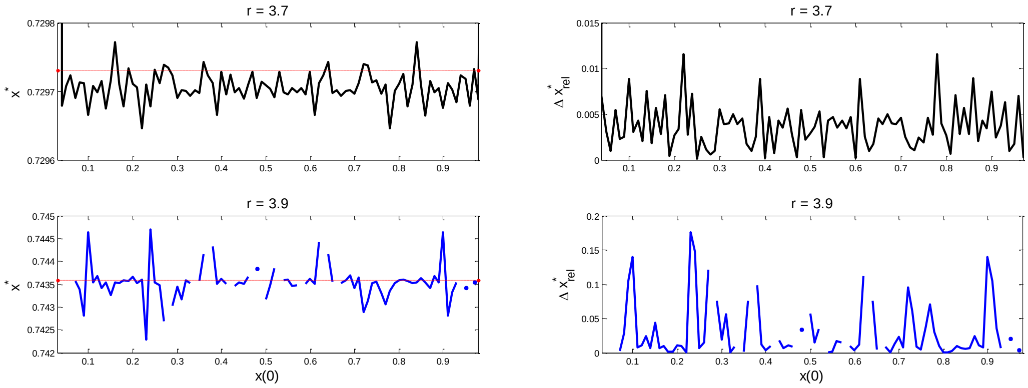

Figure 4. The fixed point x^* and its relative error Δx^*_{rel} as a function of the starting point x(0) for r = 3.7 and r = 3.9. The time series length was N = 100. The red dashed lines indicate the theoretical values $x^*_{th} = 0.72973$ (for r = 3.7), respectively $x^*_{th} = 0.74359$ (for r = 3.9)

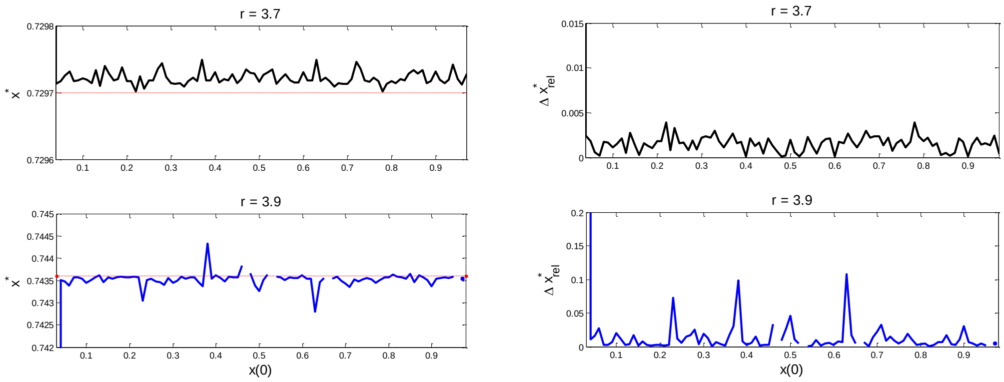

Figure 5. The same as in Figure 4 but for N = 200

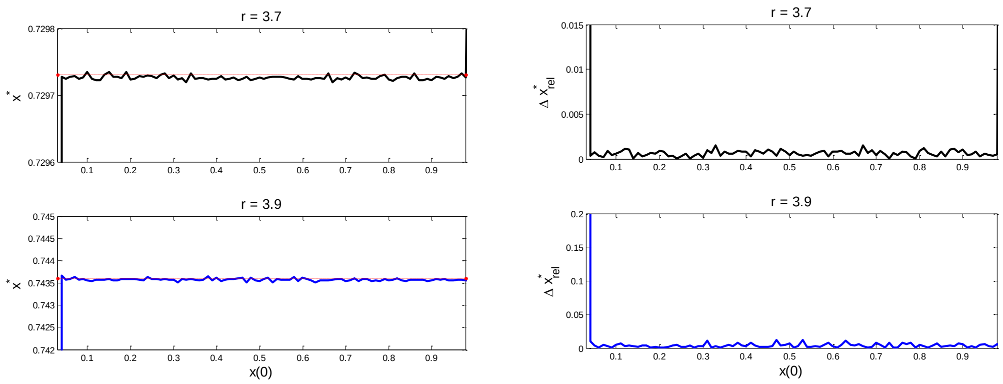

Figure 6. The same as in Figure 4 but for N = 400

The qualitative conclusions outlined above for the fixed point can be maintained for the control parameter α, as shown in Figures 7 to 9. For N = 100, the relative error $\Delta\alpha_{rel} = \left|\frac{\alpha_{sim} - \alpha_{th}}{\alpha_{th}}\right| \cdot 100\%$ is at most 1 – 2 percentages for r = 3.7 but it can increase to 10 – 20 percentages for r = 3.9 (in those cases where the equation (6) can be applied).

The situation changes in a favorable manner by increasing the number of iterations N, the same error being reduced to the maximum 0.5% for r = 3.7 and 2 % for r = 3.9.

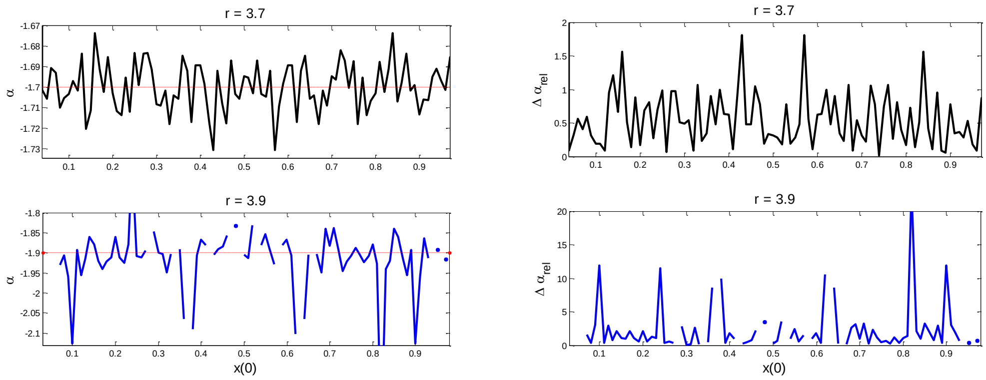

Figure 7. The control parameter α *and its relative error* $\Delta\alpha_{rel}$ *as a function of the starting point x(0) for r = 3.7 and r = 3.9. The time series length was N = 100. The red dashed lines indicate the theoretical values* $\alpha_{th} = -1.7$ *(for r = 3.7), respectively* $\alpha_{th} = -1.9$ *(for r = 3.9)*

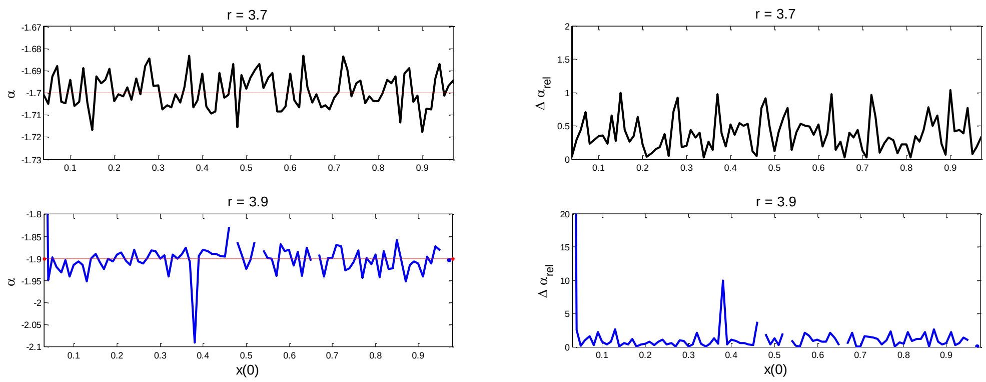

Figure 8. The same as in Figure 7 but for N = 200

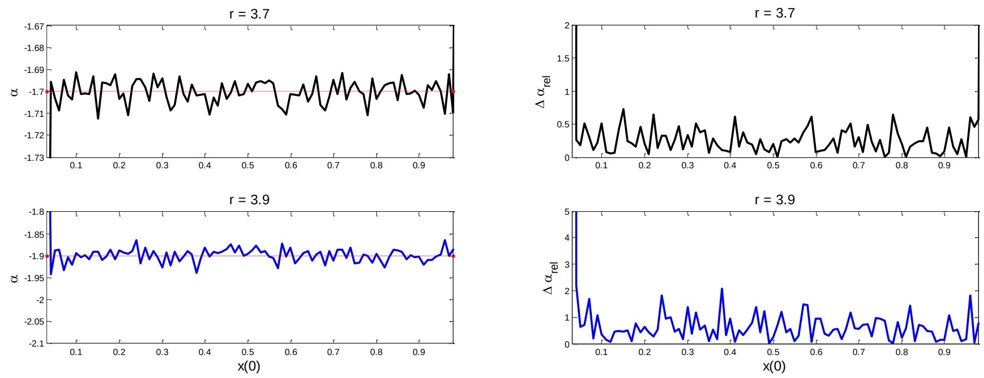

Figure 9. The same as in Figure 7 but for N = 400

The other control parameter, β, was estimated from equation (7) using $r_1 = 3.71$, respectively $r_1 = 3.91$. If N = 100, for most initial points x_0 the relative error $\Delta\beta_{rel} = \left|\frac{\beta_{sim} - \beta_{th}}{\beta_{th}}\right| \cdot 100\%$ was found to be unacceptable high, as reported in Figures 10 to 12. Just by increasing the number of iterates to 400 the error $\Delta\beta_{rel}$ decreased to maximum 25%. This observation can be explained by the fact no more values r_1 have been used.

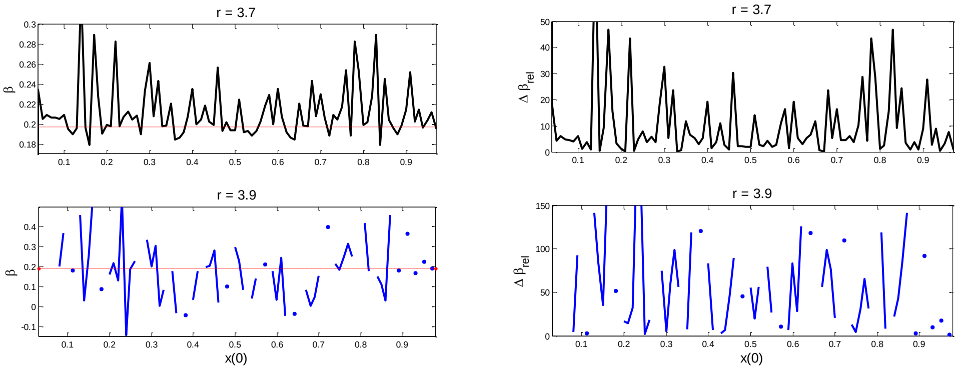

Figure 10. The control parameter β and its relative error $\Delta\beta_{rel}$ as a function of the starting point x(0) for r = 3.7 and r = 3.9. The time series length was N = 100. The red dashed lines indicate the theoretical values $\beta_{th} = 0.1972$ (for r = 3.7), respectively $\beta_{th} = 0.1907$ (for r = 3.9)

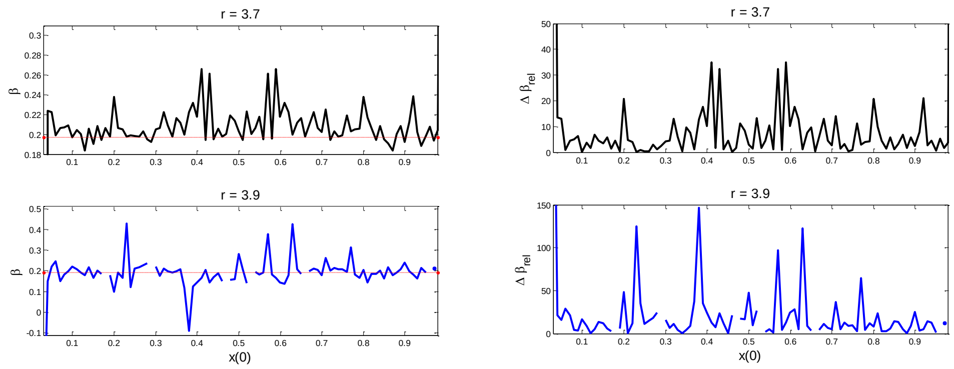

Figure 11. The same as in Figure 10 but for N = 200

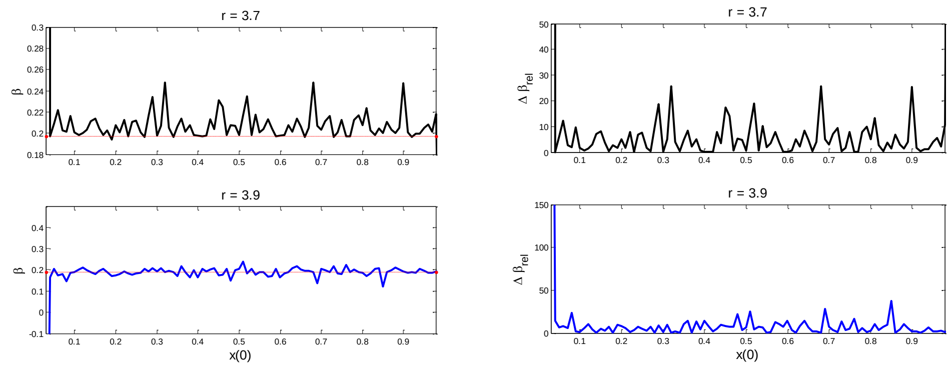

Figure 12. The same as in Figure 10 but for N = 400

The consequence of obtaining a numerical value away from the theoretical one was reflected in the γ value, as illustrated in Figure 13. To improve the value of β we could try either to change the radius ε for a close encounter or to increase the number of r_1 values in equation (7).

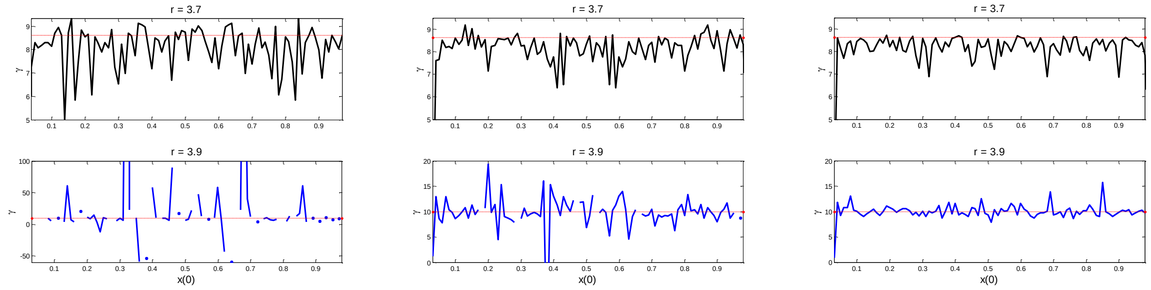

Figure 13. The control parameter γ as a function of the starting point x(0) for r = 3.7 and r = 3.9. The time series length was N = 100 (left), N = 200 (middle) or N = 400 (right). The red dashed lines indicate the theoretical values $\gamma_{th}=8.62$ (for r = 3.7), respectively $\gamma_{th}=9.97$ (for r = 3.9)

The effect of changing ε is not clear at all. Reducing the radius ε will immediately result in a lower K value, with the possibility that the method becomes inapplicable even for large time series. Increasing the radius will result in the loss of close encounter's significance. We tried three variants, namely $\varepsilon \in \{0.05, 0.1, 0.2\}$. For $r = 3.7$ the best results were obtained for $\varepsilon = 0.05$, while for $r = 3.9$ a radius of 0.1 produced the same effect (see Figures 14 and 15).

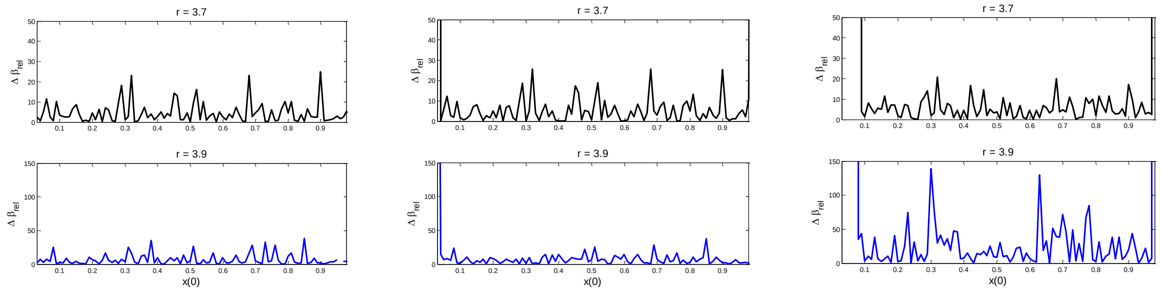

Figure 14. The relative error $\Delta\beta_{rel}$ as a function of the starting point x(0) for r = 3.7 and r = 3.9. The time series length was N = 400. The radius for a close encounter was chosen to be $\varepsilon = 0.05$ (left), $\varepsilon = 0.1$ (middle) or $\varepsilon = 0.2$ (right)

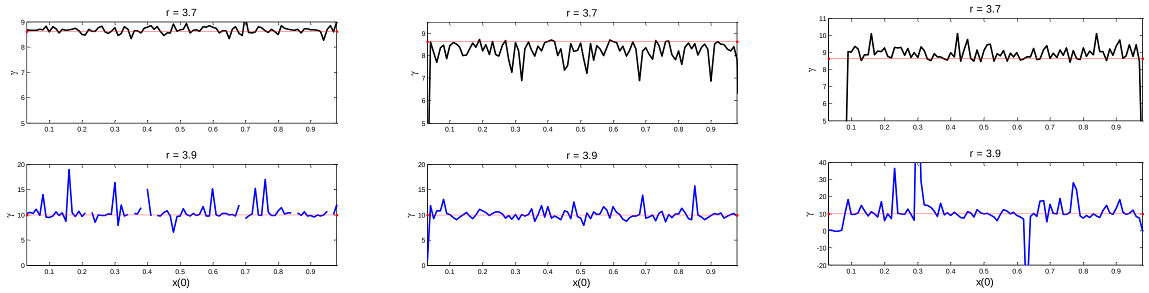

Figure 15. The control parameter γ as a function of the starting point x(0) for r = 3.7 and r = 3.9. The time series length was N = 400. The radius for a close encounter was chosen to be $\varepsilon = 0.05$ (left), $\varepsilon = 0.1$ (middle) or $\varepsilon = 0.2$ (right)

Finally, increasing the number of values r_1 leads to a value β_{sim} much closer to its theoretical counterpart, as shown in Figure 16.

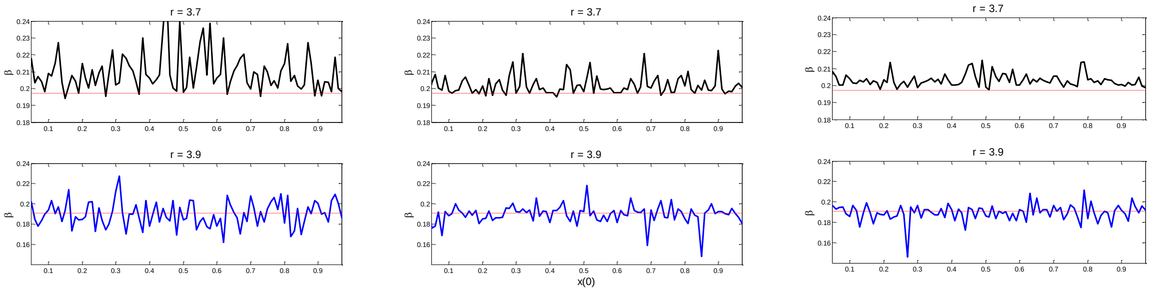

Figure 16. The control parameter β as a function of the starting point x(0) for r = 3.7 and r = 3.9. The time series length was N = 400. The number of r_1 values were one (left), three (middle) or five (right)

4. Conclusions

The paper investigates numerically the capacity of an algorithm recently proposed by Santos and Graves to accurately estimate from a time series the chaos control parameters required by OGY method. The algorithm was tested on the case of logistic map and the aim was to stabilize the fixed point. Although not presented in the paper, similar results have been found for other one-dimensional maps.

The main conclusions of the study are as follows:

a) The algorithm can be applied only if a reasonable number of close encounters of the fixed point exists. In the middle of the heavily chaotic region (r = 3.9), there are many initial conditions for which the system evolves away from the fixed point, at least for a short time series. If parameter r is chosen closer to the beginning of the chaos (for example, r = 3.7) then, regardless of the starting condition, in the vicinity of the fixed point will be found enough close encounters;

b) The fixed point was estimated with sufficient accuracy for both r values. For short time series (100 values) the relative errors do not exceed 0.015% for r = 3.7 and 0.18% for r = 3.9. In the second case, the algorithm did not work for about 10% of the starting points. This shortcoming disappeared with the increase of the time series length. For N = 400, the fixed point was estimated with extremely high precision;

c) The previous conclusion can be broadly maintained for the first control parameter, α . For short time series the relative errors are at most 1 – 2 % for r = 3.7 and 10 % for r = 3.9, but falls below 0.5 % if r = 3.7 or 2 % if r = 3.9 for a larger data set;

d) In the variant used by the authors, larger errors occur in estimating the second control parameter, β . The explanation lies in the fact that only one term was used to obtain the mean value in equation (7). Averaging over three or five values, the errors have dropped significantly, especially for large time series (e.g. N = 400).

As a final conclusion, using just two small time series involves assuming a high risk in estimating the fixed point and the control parameters for the OGY method. It is just a matter of luck in choosing the starting point if the algorithm will work or if the errors will be acceptable or not.

The algorithm can be adapted for estimating the periodic UPOs of multi-dimensional maps.

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