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HE'S HAMILTONIAN APPROACH FOR THE GENERALIZED DUFFING CONSERVATIVE OSCILLATOR

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Abstract: *The Duffing-like oscillators have been extensively applied to represent many physically systems especially in engineering sciences. Our paper is aimed to use the He's Hamiltonian approach (HA for short) for obtaining a simple analytical solution to the generalized Duffing conservative oscillator, where the restoring force is written as an odd polynomial of arbitrary degree.The HA provides also a fast and reliable estimation of frequency – amplitude relationship. Three illustrative particular cases, for which the closed-form solutions are available, are given to check the efectiveness of the HA and the accuracy of the obtained results. They correspond to the classic softening oscilator, to a simple pendulum mounted on a rotating rigid frame and to a cubic – septic Duffing oscillator, respectively. The analytial results are contrasted with their exact or numerical counterpartsand they reveal an excellent agreement for small amplitudes, acceptable discrepancies for medium amplitudes and high enough relative errors for large oscillation amplitudes, when the oscillator behaves unharmonically. For the simple case of the softening cubic oscillator, an improved approximation is derived too.*

Keywords: *Duffing equation, Hamiltonian approach, Approximate solution.*

INTRODUCTION

Starting with their introduction in science by the german engineer Georg Duffing (1918),Duffing-like oscillators have received remarkable attention because of the huge number of their engineering applications. A short list of the phenomena modeled by the nonlinear Duffing equation may include the vibrations of beams and plates, the large amplitude oscillations of centrifugal governor systems, the vibrations induced on different structures by fluid flow, the oscillations of pendulum-like systems or of magneto-elastic mechanical systems.

The Duffing equation is frequently cited in a form containing a linear viscous damping, a cubic restoring force and a forcing term with a single-frequency excitation. This is the shape used first by Duffing to explain forced vibrations of industrial machinery having linear damping [1]. Despite its apparent simplicity, the equation has many catastrophic, diverging and oscillative behaviors by several stable/ unstable states obtained by changing the coefficients [2].

In the paper we consider a generalized Duffing equation, where the restoring force is written as an odd polynomial of any degree. Instead, the damping and forcing terms are missing giving us the conservative oscillator

$$\ddot{x}+\alpha_1 x+\alpha_3 x^3+...+\alpha_{2n-1}x^{2n-1}=0$$
(1)

with the initial conditions $x(0)=A$ and $\dot{x}(0)=0$.

Surveying the literature shows that some particular cases of equation (1) have been studied by several techniques, including He's variational approach [3], He's energy balance method [4], homotopy perturbation method [5] and frequency-amplitude formulation [6].

Our motivation in the present study is to obtain the frequency-amplitude relationship and the approximate solution of the Duffing equation (1) by the He's Hamiltonian approach and to compare the results with those proided by their exact or numerical counterparts on some relevant particular cases.

HAMILTONIAN APPROACH. BASIC IDEA

Consider the following equation which describes the motion of a single-degree-of-freedom conservative nonlinear oscillator

$$\ddot{x}+f(x)=0,\ x(0)=A,\ \dot{x}(0)=0 \quad (2)$$

where *x(t)* is the dependent variable and*t* is the time. As usual, $\ddot{x}$ represents the second time derivative d^2x/dt^2. There exist an associated Hamiltonian

$$H=\frac{1}{2}\dot{x}^2+F(x)=F(A) \quad (3)$$

with $F(x)=\int f(x)dx$. According to He's Hamiltonian approach [7] a new function is introduced as

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$\tilde{H}(x)=\int_0^{T/4}\left(\frac{1}{2}\dot{x}^2+F(x)\right)dt=\frac{T}{4}\cdot H$ = constant (4)

where $T=2\pi/\omega$ and ω is the unknown angular frequency of the motion.
A solution for equation (2) that satisfies the initial conditions is assumed as

$$x(t)=A\cos\omega t$$
(5)

By introducing it into the equation (4), after computing the integral one has $\tilde{H}(A,\omega)=\frac{T}{4}\cdot H$, so $\partial\tilde{H}/\partial T=H/4=$constant. The dependence of angular frequency ω on amplitude A results now from the equation $\frac{\partial}{\partial A}\left(\frac{\partial\tilde{H}}{\partial T}\right)=0$ or, equivalently

$$\frac{\partial}{\partial A}\left(\frac{\partial\tilde{H}}{\partial(1/\omega)}\right)=0$$
(6)

APPLICATION TO THE GENERALIZED DUFFING EQUATION

The Hamiltonian of equation (1) is as follows

$$H=\frac{1}{2}\dot{x}^2+\frac{\alpha_1}{2}x^2+\frac{\alpha_3}{4}x^4+...+\frac{\alpha_{2n-1}}{2n}x^{2n}$$
(7)

Considering the trial solution $x(t)=A\cos\omega t$ and integrating (7) with respect to time from 0 to $T/4$, one gets

$$\tilde{H}(A,\omega)=\int_0^{T/4}\left(\frac{A^2\omega^2}{2}\sin^2\omega t+\frac{\alpha_1A^2}{2}\cos^2\omega t+\right.$$

$$\left.+\frac{\alpha_3A^4}{4}\cos^4\omega t+...+\frac{\alpha_{2n-1}A^{2n}}{2n}\cos^{2n}\omega t\right)dt.$$

But $\int_0^{T/4}\sin^2\omega t\,dt\overset{\tau=\omega t}{=}\frac{1}{\omega}\int_0^{\pi/2}\sin^2\tau\,d\tau=\frac{\pi}{4\omega}$

and

$$\int_0^{T/4}\cos^{2n}\omega t\,dt\overset{\tau=\omega t}{=}\frac{1}{\omega}\int_0^{\pi/2}\cos^{2n}\tau\,d\tau=\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2\omega},$$

so $\tilde{H}(A,\omega)=\frac{\pi\omega}{8}A^2+\sum_{k=1}^{n}\frac{(2k-1)!!\alpha_{2k-1}\pi}{4k\cdot(2k)!!\omega}A^{2k}$. It follows that

$$\frac{\partial\tilde{H}}{\partial(1/\omega)}=-\frac{\pi\omega^2}{8}A^2+\sum_{k=1}^{n}\frac{(2k-1)!!\alpha_{2k-1}\pi}{4k\cdot(2k)!!}A^{2k}$$

and

$$\frac{\partial}{\partial A}\left(\frac{\partial\tilde{H}}{\partial(1/\omega)}\right)=-\frac{\pi\omega^2}{4}A+\sum_{k=1}^{n}\frac{(2k-1)!!\alpha_{2k-1}\pi}{2(2k)!!}A^{2k-1}.$$

Setting $\frac{\partial}{\partial A}\left(\frac{\partial\tilde{H}}{\partial(1/\omega)}\right)=0$ results in frequency – amplitude relationship

$$\omega_{appx}=\sqrt{\alpha_1+\frac{3\alpha_3}{4}A^2+\frac{15\alpha_5}{24}A^4+...+\frac{2(2n-1)!!\alpha_{2n-1}}{(2n)!!}A^{2n-2}}$$
(8)

Substituting equation (8) into (5) yields the approximate solution for the Duffing equation (1)

$$x_{appx}(t)=A\cos\omega_{appx}t$$
(9)

PARTICULAR CASES

In this section, the Hamiltonian approach is applied to three particular cases of Duffing oscillators and the results are contrasted with their numerical and exact counterparts.

Example 1: The first example concerns the softening Duffing oscillator

$$\ddot{x}+x-x^3=0,\ x(0)=A,\ \dot{x}(0)=0$$
(10)

for which a closed-form exact solution is available. It is expressed in terms of Jacobi *sn* elliptic function

$$x_{exact}(t)=A\,sn\left(\left(1-\frac{A^2}{2}\right)t,\frac{a}{\sqrt{2-A^2}}\right)$$
(11)

and has the period

$$T_{exact}=\frac{4K\left(A/\sqrt{2-A^2}\right)}{\sqrt{1-A^2/2}}$$
(12)

with $K(k)$ denoting the complete elliptic integral of first kind [8]. Equation (10) describes, for example, the unforced, undamped ship capsizing [9]. Because $\alpha_1=1,\alpha_3=-1,\alpha_{2n-1}=0$ for all $n\geq3$ one has for the approximate frequency the expression

$$\omega_{appx}=\sqrt{1-\frac{3}{4}A^2} \quad (13)$$

Despite its simplicity, expression (13) provides an excellent approximation for the dependence of oscillation frequency on amplitude, at least for amplitudes smaller than 0.75. The relative error,

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computed with $ERR = 100 \cdot \left(\omega_{exact} - \omega_{appx}\right) / \omega_{appx}$, does not exceed 1%. Starting with *A* = 0.75, the differences between exact frequency and the approximate one increase rapidly and overcome 10% for *A* = 0.95, as shown in Table 1 and Figures 1 and 2.

Table 1: The dependence of exact and approximate frequencies on amplitude

A	ω_{exact}	ω_{appx}	*ERR* (%)
0.1	0.9967	0.9966	0.01
0.3	0.9655	0.9657	0.02
0.5	0.9007	0.9014	0.08
0.7	0.7953	0.7899	0.68
0.9	0.5923	0.6265	5.77
0.95	0.5107	0.6265	11.3

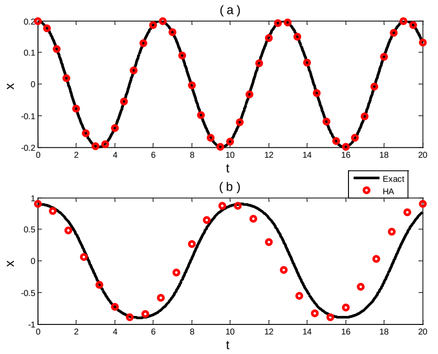

Figure 1. The comparison between the time series solutions x = x(t) obtained with exact and approximate formulas (11) and (9). The red circles stand for Hamiltonian approach solution.
a) A = 0.2; b) A = 0.9.

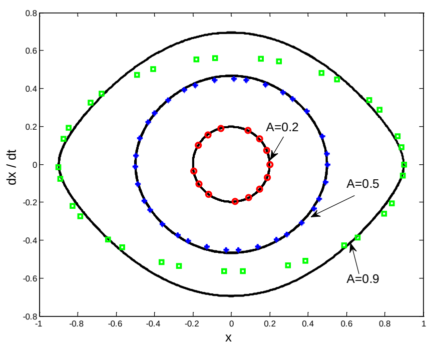

Figure 2. The phase plane (x, dx/dt) corresponding to Duffing oscillator (10) for different values of amplitudes A. The continuous dark lines stand for exact solution while the colored circles represent Hamiltonian approach solution

It is possible to construct an improved approximation for large amplitudes *A* considering a solution of the form

$$x_{appx}^{impr}(t) = a\cos(\omega_{appx}^{impr} \cdot t) + b\cos\left(3\omega_{appx}^{impr} \cdot t\right)$$
(14)

with $a + b = A$. Replacing it into the function $\tilde{H}$, one has

$$\tilde{H}(a,b,\omega) = \int_0^{T/4}\left[\frac{1}{2}\left(-a\omega\sin\omega t - 3b\omega\sin 3\omega t\right)^2 + \frac{\left(a\cos\omega t + b\cos 3\omega t\right)^2}{2} - \frac{\left(a\cos\omega t + b\cos 3\omega t\right)^4}{4}\right]dt$$

After computations, one gets

$$\tilde{H}(a,b,\omega) = \frac{\pi}{8\omega}\left(a^2\omega^2 + 9b^2\omega^2 + a^2 + b^2 - \frac{3a^4}{8} - \frac{a^3 b}{2} - \frac{3a^2 b^2}{2} - \frac{3b^4}{8}\right)$$

. This time the frequency-amplitude relationship results by solving the system $\frac{\partial}{\partial a}\left(\frac{\partial \tilde{H}}{\partial(1/\omega)}\right) = 0$, $\frac{\partial}{\partial b}\left(\frac{\partial \tilde{H}}{\partial(1/\omega)}\right) = 0$. It becomes

$$\begin{cases} -\omega^2 + 1 - \frac{3a^2}{4} - \frac{3ab}{4} - \frac{3b^2}{2} = 0 \\ -9b\omega^2 + b - \frac{a^3}{4} - \frac{3a^2 b}{2} - \frac{3b^3}{4} = 0 \end{cases}.$$

From its first equation we derive the desired approximation for the oscillation's frequency

$$\omega_{appx}^{impr} = \sqrt{1 - \frac{3a^2}{4} - \frac{3ab}{4} - \frac{3b^2}{2}}$$
(15)

Using $b = A - a$ and solving the second equation of the above system with respect to *a* one gets the following third order polynomial

$$46a^3 - 120Aa^2 + \left(126A^2 - 32\right)a + \left(32A - 51A^3\right) = 0$$

It can be solved for different *A* and the acceptable solutions for *a* are then used to find *b*, ω_{appx}^{impr} and the function $x_{appx}^{impr}(t)$. Some relevant results are displayed in Table 2 and Figure 3. It is obvious that the improved algorithm with two terms in the approximate solution, $x_{appx}^{impr}(t)$, gives much better results for large amplitudes of oscillation.

Table 2: The dependence of exact and approximate improved frequencies on amplitude

A	*a*	ω_{exact}	ω_{appx}^{impr}	*ERR* (%)
0.5	0.5048	0.9007	0.9004	0.03

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0.6	0.6093	0.8521	0.8519	0.02
0.7	0.7171	0.7899	0.7894	0.06
0.8	0.8317	0.7087	0.7067	0.28
0.9	0.9644	0.5923	0.5855	1.15
0.95	1.0576	0.5107	0.4786	6.26

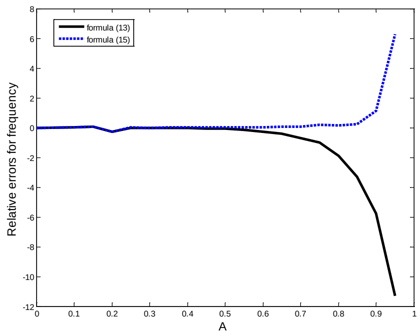

Figure 3. The increase with amplitude A of relative errors between exact frequency ω_{exact} and its approximate counterparts ω_{appx} and ω_{appx}^{impr}.

Example 2: In what follows, we apply the Hamiltonian approach to the cubic-quintic Duffing oscillator

$$\ddot{x}+\alpha_1 x+\alpha_3 x^3+\alpha_5 x^5=0,\ x(0)=A,\ \dot{x}(0)=0 \quad (16)$$

The equation (16) can be found in the modeling of different physical phenomena, including the free vibration of a restrained uniform beam carying intermediate lumped mass and undergoing large amplitudes of oscillation, the propagation of a short electromagnetic pulse in a nonlinear medium or the oscillations of pendulum-like systems.
There exist an exact but complicated frequency-amplitude relationship [10], having the form

$$\omega_{exact}=\frac{\pi k_1}{2\int_0^{\pi/2}\left(1+k_2\sin^2 t+k_4\sin^4 t\right)^{-1/2}dt} \quad (17)$$

with

$$k_1=\sqrt{\alpha_1+\frac{\alpha_2 A^2}{2}+\frac{\alpha_3 A^4}{3}},\ k_2=\frac{3\alpha_3 A^2+2\alpha_5 A^4}{6\alpha_1+3\alpha_3 A^2+2\alpha_5 A^4}$$

$$k_2=\frac{2\alpha_5 A^4}{6\alpha_1+3\alpha_3 A^2+2\alpha_5 A^4}$$

(18)

As we will see, it can be well approximated by equation (8) that customizes as

$$\omega_{appx}=\sqrt{\alpha_1+\frac{3\alpha_3}{4}A^2+\frac{5\alpha_5}{8}A^4}$$

(19)

The application of the above expression will be made on the particular case of rotating simple pendulum, that is described in brief below.

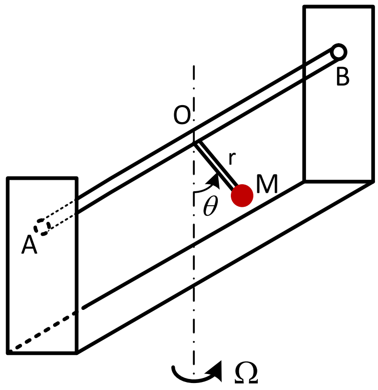

Figure 4. A simple pendulum mounted on a rotating rigid frame

A rigid frame is forced to rotate at the constant angular frequency Ω [11]. While the frame rotates, a simple pendulum characterized by length *r* and mass *m* oscillates (see Figure 4). Assuming that the bar ABM is weightless, the pendulum's governing equation of motion is

$$\theta''+(1-\Lambda\cos\theta)\sin\theta=0,\theta(0)=A,\ \theta'(0)=0 \quad (20)$$

where $\Lambda=\frac{\Omega^2 r}{g}$ and the derivatives are taken with respect to the non-dimensional time $\bar{t}=\sqrt{\frac{g}{r}}t$. The equation (20) could be rewritten as

$$\theta''+\sin\theta-\frac{\Lambda}{2}\sin 2\theta=0 \quad (21)$$

But $\sin\varphi\cong\varphi-\frac{\varphi^3}{6}+\frac{\varphi^5}{120}$, so we can approximate the rotating pendulum motion by using the equation

$$\theta''+(1-\Lambda)\theta+\left(\frac{2\Lambda}{3}-\frac{1}{6}\right)\theta^3+\left(\frac{1}{120}-\frac{2\Lambda}{15}\right)\theta^3=0$$

that is a particular case of (16) with

$$\alpha_1=1-\Lambda,\ \alpha_3=\frac{2\Lambda}{3}-\frac{1}{6},\ \alpha_5=\frac{1}{120}-\frac{2\Lambda}{15} \quad (22)$$

In conclusion, the Hamiltonian approach gives for the rotating pendulum problem the following frequency-amplitude relationship

$$\omega_{appx}=\sqrt{1-\Lambda+\left(\frac{\Lambda}{2}-\frac{1}{8}\right)A^2+\left(\frac{1}{192}-\frac{\Lambda}{12}\right)A^4} \quad (23)$$

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The parameter Λ is proportional to frequency Ω. There are physically acceptable pairs (A,Λ) for which $\omega_{appx}^2 > 0$ and pairs (A,Λ) so the opposite conclusion is valid. A plot of ω_{appx}^2 values for $A \in [0,3]$ and $\Lambda \in [0,5]$ is shown in Figure 5. The dark curve signifies the border between real and imaginary values of ω_{appx}. It is evident that at least for the combinations (A,Λ) in the upper area the expression for ω_{appx} provided by the Hamiltonian approach ceases to represent a reliable approximation for the exact solution.

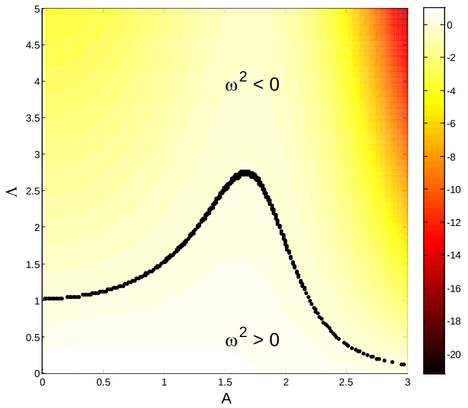

Figure 5.A plot of ω_{appx}^2 values as a function of amplitude of oscillation, A, and parameter Λ.

For most pairs (A,Λ) with $\omega_{appx}^2 > 0$ (asfigures 6 and 7, respectively Tables 3 and 4, prove) the function $\theta_{appx} = A\cos\omega_{appx} t$ represents a trustworthy approximation for the exact solution of problem (21) even for large amplitude of oscillations (close to $2\pi/3$). The relative errors increase with Λ, but they remain reasonably small.

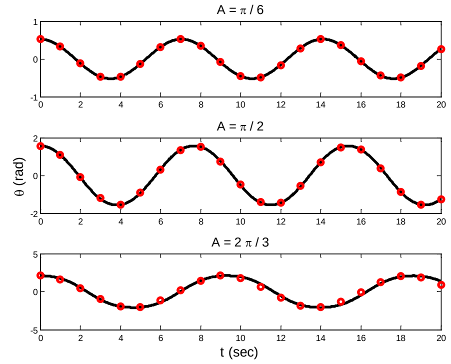

Figure 6. Comparison of Hamiltonian approach solution (red circles) and numerical solution (continuous black curve) for $\Lambda = 0.2$ and different amplitudes A

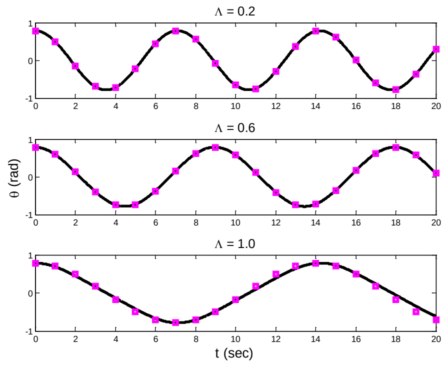

Figure 7. Comparison of Hamiltonian approach solution (magenta dots) and numerical solution (continuous curve) for $A = \pi/4$ and different Λ

Table 3: Numerical versus approximate periods if $\Lambda = 0.2$

A	T_{num}	T_{appx}	*ERR* **(%)**
$\pi/12$	7.0321	7.0360	0.06
$\pi/6$	7.0558	7.0589	0.04
$\pi/4$	7.1101	7.1133	0.05
$\pi/3$	7.2096	7.2130	0.06
$\pi/2$	7.6924	7.6844	0.10
$2\pi/3$	9.3486	9.1663	1.95
$3\pi/4$	13.7008	11.3200	17.37

Table 4: Numerical versus approximate periods if $\Lambda = 1.0$

A	T_{num}	T_{appx}	*ERR* **(%)**
$\pi/12$	40.3285	39.4747	2.012
$\pi/6$	20.6006	20.1807	2.04

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$\pi/4$	14.2573	13.9940	1.85
$\pi/3$	11.3292	11.1547	1.54
$\pi/2$	9.4003	9.3701	0.32

However, there are pairs (A,Λ) located just below the dark line in Figure 5 for which the exact solution behaves completely different from the approimate one. As an example, for $\Lambda = 0.2$ and $A = 3\pi/4$ the graph of the numerical solution presents almost horizontal sectors near the extremes (the upper part of Figure 8). For $(A,\Lambda) = (\pi/2, 2.0)$ or $(A,\Lambda) = (\pi/6, 1.1)$ the numerical solutions comport like *sine* functions but with a smaller period (compared with the approximate solution) and with another center of symmetry (see middle and lower parts of Figure 8).

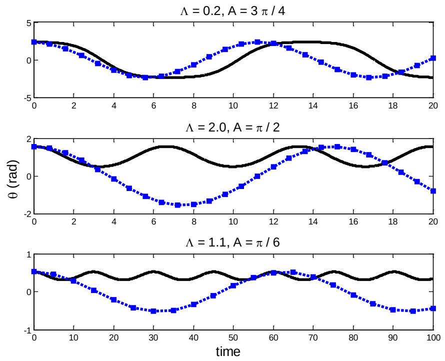

Figure 8. Comparison between Hamiltonian approach solution and its numerical counterpart that indicates notable differences between them.

Example 3: We close the study by considering the highly nonlinear Duffing equation

$$\ddot{x} + x^3 + x^7 = 0,\ x(0) = A,\ \dot{x}(0) = 0$$
(24)

The treatment of oscillator ruled by equation (24) is elementary, but not simple [12]. All the solutions are periodic and involve Gauss and Appell hypergeometric functions in their closed form expression. The period of oscillations depends on amplitude according to the law

$$T(A) = \sqrt{2\pi}\,\frac{\Gamma(1/4)}{\Gamma(3/4)}\left(\frac{1}{A} - \frac{A^3}{3} + \frac{5A^7}{28} - \frac{5A^{11}}{4} + \ldots\right)$$

represented a truncated series of the exact solution. The Hamiltonian approach gives for the same quantity the result

$$T_{appx}(A) = 2\pi \Big/ \sqrt{\frac{3}{4}A^2 + \frac{35}{64}A^6}$$
(25)

For small amplitudes *A*, the numerical and Hamiltonian approach solutions agree fairly well, as shown in Figure 9. The curve *x*= *x*(*t*) preserves the shape of a sinusoid with some tendency to drill the tips. On contrary, the derivative curve d*x* / d*t* has blunt extremes.

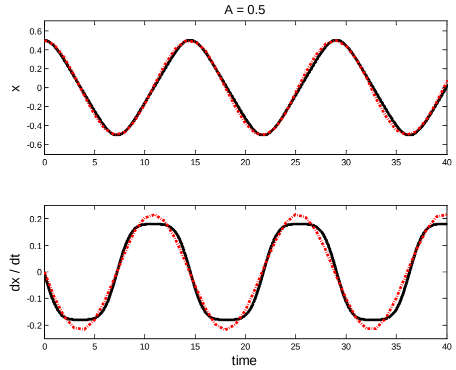

Figure 9. Comparison between Hamiltonian approach solution (dashed red curve) and its numerical counterpart for A = 0.5.

For large *A*, the differences between the two types of solutions become enormous, both in terms of period and shape, as illustrated in Figure 10. Even though both methods indicate a decrease in the period with increasing amplitude, the rate of change is much larger for Hamiltonian approach. Furthermore, the saw teeth shape of curve *x* = *x*(t) is more pronounced as well as the flattening of curve d*x* / dt.

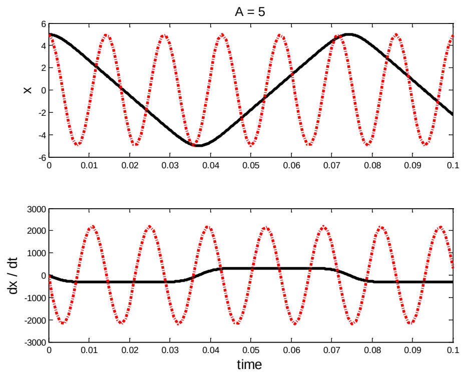

Figure 10. Comparison between Hamiltonian approach solution (dashed red curve) and its numerical counterpart for A = 5.

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Conclusions

The purpose of the paper was to apply He's Hamiltonian approach (HA for short) to the generalized Duffing conservative oscillator for finding the dependence of the frequency on its amplitide and for providing a reliable approximation for its exact solution. The general results were particularized for the cubic softing oscillator, the rotating pendulum and the cubic-septic oscillator. All the examples show an excellent agreement between the HA and numerical/ exact solution, at least for small and intermediate amplitudes of oscillation. We conclude that HA is a simple and efficient method for approximating periodic solutions of nonlinear generalized Duffing oscillator.

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