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# $(h,\varphi)-$OPTIMALITY CONDITIONS FOR LOCALLY LIPSCHITZ GENERALIZED B-VEX SEMI-INFINITE PROGRAMMING

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***Abstract:*** *In this paper, by using $(h,\varphi)-$ generalized directional derivative and $(h,\varphi)-$ generalized gradient, the class of B-vex, $(\rho,B,\eta)$-invex, pseudo $(B,\eta)$-invex, and quasi $(B,\eta)$-invex functions for differentiable functions is extended to the class of generalized $(h,\varphi)-$ B-vex, $(h,\varphi)-(\rho,B,\eta)-$invex, pseudo $(h,\varphi)-(B,\eta)-$invex, and quasi $(h,\varphi)-(B,\eta)-$invex functions for locally Lipschitz functions. The sufficient optimality conditions are obtained for semi-infinite programming problems which involving those functions.*

***Keywords:*** *$(h,\varphi)-$ generalized directional derivative, $(h,\varphi)-$ generalized gradient ,locally Lipschitz function, generalized $(h,\varphi)-$ B-vex.*

## 1. Introduction

It is well known that convexity play an important role in establishing the sufficient optimality conditions and duality theorems for a nonlinear programming problem. Several class of functions have been defined for the purpose of weakening the limitations of convexity. Bector and Singh extend the class of convex functions to the class of B-vex functions in [2]. In [3], Bector and Suneja define the class of B-invex functions for differentiable numerical functions. The sufficient optimality conditions and duality results were obtained involving these generalized functions. As so far now, the study about the sufficient optimality conditions and algorithm of semi-infinite programming are under the assumption that the involving functions are differentiable. But non-smooth phenomena in mathematics and optimization occur naturally and frequently, and there is a need to be able to deal with them. In [15], the author study some of the properties of B-vex functions for locally Lipschitz functions, and extend the class of B-invex, pseudo B-invex and quasi B-invex functions from differentiable numerical functions to locally Lipschitz functions. In [11-13], Preda introduced some classes of V-univex type-I functions , called called ($\rho$, $\rho$')-V-univex type-I, ($\rho$, $\rho$')-quasi V-univex type-I, ($\rho$, $\rho$')-pseudo V-univex type-I, ($\rho$, $\rho$')-quasi pseudo V-univex type-I, and ($\rho$, $\rho$')-pseudo quasi V-univex type-I. In [8] Preda introduced the class of locally Lipschitz (B, ρ ,d ) -preinvex functions and extend many results of B-vexity type stated in literature. Preda introduced $(F,\rho)-$convex function as extension of *F*−convex function and ρ−convex function[9,10,14].

The sufficient optimality conditions and duality results are obtained for nonlinear programming, generalized fractional programming, multi-objective programming and minmax programming problem, which involving those functions.

Ben Tal [4] obtained some properties of $(h,\varphi)-$convex functions based on the above generalized algebraic operations. These results were also applied to some problems in statistical decision theory.

In this paper by using $(h,\varphi)-$ generalized directional derivative and $(h,\varphi)-$ generalized gradient [6], the class of B-vex, B-invex, pseudo B-invex, and quasi B-invex functions for differentiable functions [16] is extended to the class of generalized $(h,\varphi)-$ B-vex, $(h,\varphi)-(\rho,B,\eta)-$invex,pseudo $(h,\varphi)-(B,\eta)-$invex, and quasi $(h,\varphi)-(B,\eta)-$invex functions for locally Lipschitz functions. The sufficient optimality conditions are obtained for semi-infinite programming problems which involving those functions.



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This paper is organized as follows. In Section 2, we present some preliminaries and related results which will be used in the rest of the paper. The definiton of B-vex, $(\rho,B,\eta)-$invex, $(B,\eta)-$pseudo-invex, $(B,\eta)-$quasi-nvex function are given in sense of $(h,\varphi)$. In Section 3, some sufficient optimality conditions theorems are derived.

## 2. Preliminares and some properties of generalized $(h,\varphi)-(B,\eta)-$invex functions

Throughout our presentation, we let $X$ be a non-empty convex subset of $R^n$, $U$ be an open unit ball in $R^n$, $R_+$ denote the set of nonnegative real numbers, $f: X\to R$, $b: X\times X\times[0,1]\to R_+$, and $b(x,u,\lambda)$ is continuous at $\lambda=0$ for fixed $x,u$.

Ben-Tal [4] introduced certain generalized operations of addition and multiplication.

1) Let *h* be an *n* vector-valued continuous function, defined on a subset $X$ of $R^n$ and possessing an inverse function $h^{-1}$. Define the *h*-vector addition of $x\in X$ and $y\in X$ as

$$x\oplus y=h^{-1}\left(h(x)+h(y)\right),$$

and the *h*-scalar multiplication of $x\in X$ and $\lambda\in R$ as

$$\lambda\otimes x=h^{-1}\left(\lambda h(x)\right).$$

2) Let $\varphi$ be a real-valued continuous functions, defined on $\Phi\subseteq R$ and possessing an inverse functions $\varphi^{-1}$. Then the $\varphi$-addition of two numbers, $\alpha\in\Phi$ and $\beta\in\Phi$, is given by

$$\alpha[+]\beta=\varphi^{-1}\left(\varphi(\alpha)+\varphi(\beta)\right),$$

and the $\varphi$-scalar multiplication of $\alpha\in\Phi$ and $\lambda\in R$ by

$$\lambda[\cdot]\alpha=\varphi^{-1}\left(\lambda\varphi(\alpha)\right).$$

3) The $(h,\varphi)$-inner product of vectors $x,y\in X$ is defined as

$$\left(x^T y\right)_{h,\varphi}=\varphi^{-1}\left(h(x)^T h(y)\right).$$

Denote

$$\overset{m}{\underset{i=1}{\oplus}} x^i = x^1\oplus x^2\oplus\ldots\oplus x^m,\quad x^i\in X,\quad i=1,2,\ldots,m$$

$$\left[\sum_{i=1}^{m}\right]\alpha_i=\alpha_1[+]\alpha_2[+]\ldots[+]\alpha_m,\quad \alpha_i\in\Phi,\quad i=1,2,\ldots,m$$

$$x\Theta y=x\oplus\left((-1)\otimes y\right)$$

$$\alpha[-]\beta=\alpha[+]\left((-1)[\cdot]\beta\right).$$

By Ben-Tal generalized algebraic operation, it is easy to obtain the following conclusions:

$$\overset{m}{\underset{i=1}{\oplus}} x^i = h^{-1}\left(\sum_{i=1}^{m} h\left(x^i\right)\right)$$

$$\left[\sum_{i=1}^{m}\right]\alpha_i=\varphi^{-1}\left(\sum_{i=1}^{m}\varphi\left(\alpha_i\right)\right)$$

$$x\Theta y=h^{-1}\left(h(x)-h(y)\right)$$

$$\alpha[-]\beta=\varphi^{-1}\left(\varphi(\alpha)-\varphi(\beta)\right)$$

$$\varphi\left(\lambda[\cdot]\alpha\right)=\lambda\varphi(\alpha)$$

$$h(\lambda\otimes x)=\lambda h(x)$$

**Definition 2.1.** A real valued function $f: X\to R$ is said to be $(h,\varphi)-$Lipschitz at $x\in X$ if there exists two positive constants $\varepsilon,k$ such that

$$\left|f(z)[-]f(y)\right|_{(h,\varphi)}\le k[\cdot]\left\|z\Theta y\right\|_{(h,\varphi)},$$

$$\forall z,y\in B_{\varepsilon,(h,\varphi)}(x)$$

$f$ is said to be $(h,\varphi)-$ locally Lipschitz if $f$ is $(h,\varphi)-$Lipschitz at every $x\in X$.

Let $f$ be a Lipschitz and real-valued function defined on $R^n$. For all $x,v\in R^n$, the $(h,\varphi)-$generalized directional derivative of $f$ with respect to direction $v$ and the $(h,\varphi)-$generalized gradient of $f$ at $x$, denoted by $f^*(x,v)$ and $\partial^* f(x)$, respectively, are defined as follows[6].

$$f^*(x,v)=\limsup_{\substack{y\to x\\ t\downarrow 0}}\frac{1}{t}[\cdot]\left(f(y\oplus t\otimes v)[-]f(y)\right),$$



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$$\partial^* f*(x)=\left\{\xi^*;\, f^*(x,v)\geq\left(\xi^{*T}v\right)_{(h,\varphi)},\ \forall v\in R^n\right\}.$$

We note that the above definitions can be seen as generalizations of the definitions introduced by Zhang [16].

The relation between $(h,\varphi)-$generalized directional derivative and Clarke directional derivative can be given by the following theorem.

**Theorem 2.1.[6].** Let $f$ be a real valued function, $\varphi(t)$ be strictly increasing and continuous on $R$, and let $\hat{f}(t)=\varphi f h^{-1}(t)$.

Then $f^*(x,v)=\varphi^{-1}\left(\hat{f}^\circ(h(x),h(v))\right)$, where $f^\circ$ is Clarke directional derivative.

**Theorem 2.2.[6].** Let $f$ be a real valued function, $\varphi(t)$ be strictly increasing and continuous on $R$, and let $\hat{f}(t)=\varphi f h^{-1}(t)$. Then

$$\partial^* f*(x)=h^{-1}\left(\partial\hat{f}(h(x))\right)=\left\{h^{-1}(\xi);\xi\in\partial\left(\hat{f}(t)\Big|_{t=h(x)}\right)\right\}.$$

**Lemma 2.1.[6]** Let $f:X\to R$, $g:X\to R$ be local Lipschitz functions. Then

(i) $\partial^* f(x)$ is a non-empty, convex and weakly compact subset of $X$;

(ii) $f^*(x,v)=\max\left\{\left(\left(\xi^{*T}v\right)_{(h,\varphi)}\right);\xi^*\in\partial^* f(x)\right\},\ \forall v\in R^n;$

(iii) $\partial^*\left(f[+]g\right)(x)\subset\partial^* f(x)\oplus\partial^* g(x)$

$\partial^*\left(\lambda[\cdot]f\right)(x)=\lambda \odot \partial^* f(x),\ \forall\lambda\geq 0;$

(iv) $\left([-]f\right)^*(x,v)=f^*(x,\Theta v),\ \forall x,v\in R^n$

$\partial^*\left([-]f\right)(x)=\Theta\partial^* f(x),\ \forall x\in R^n;$

**Lemma 2.2.[6]** If $\Omega_1$ and $\Omega_2$ are two non-empty convex weakly compact of $R^n$, then

(i) for every $v\in R^n$, $\lambda_1,\lambda_2\in R$, we have

$$\max\left\{\left(\xi^{*T}v\right)_{(h,\varphi)};\xi^*=\lambda_1\otimes\xi^*{}_1\oplus\lambda_2\otimes\xi^*{}_2\right\}$$
$$=\lambda_1[\cdot]\max\left\{\left(\xi^*{}_1{}^T v\right)_{(h,\varphi)};\xi^*{}_1\in\Omega_1\right\}[+]$$
$$\lambda_1[\cdot]\max\left\{\left(\xi^*{}_2{}^T v\right)_{(h,\varphi)};\xi^*{}_2\in\Omega_2\right\}$$

(ii) $\Omega_1\subset\Omega_2$ iff for every $v\in R^n$, we have

$$\max\left\{\left(\xi^{*T}v\right)_{(h,\varphi)};\xi^*\in\Omega_1\right\}\leq\max\left\{\left(\xi^{*T}v\right)_{(h,\varphi)};\xi^*\in\Omega_2\right\}.$$

In the following definitions, we consider function $b:X\times X\times[0,1]\to R_+$, with $b(x,u,\lambda)$ continuous at $\lambda=0$ for fixed $x,u$.

We introduce following classes of functions:

**Definition 2.2.** The function $f$ is said to be $(h,\varphi)-B-$vex at $u\in X$ if for $0\leq\lambda\leq 1$ and every $x\in X$ we have

$$f\left(\lambda\otimes x\oplus(1-\lambda)\otimes u\right)\leq$$
$$\lambda b(x,u,\lambda)[\cdot]f(x)[+](1-\lambda b(x,u,\lambda))[\cdot]f(u)\quad(1)$$

$f$ is said to be $(h,\varphi)-B-$vex on $X$ if it is $(h,\varphi)-B-$vex at every $u\in X$.

**Lema 2.3.** A locally Lipschitz function $f:X\to R$ is said to be $(h,\varphi)-B-$vex with respect to $\overline{b}$ at $u\in X$ iff there exists a function $b(x,u,\lambda)$ such that

$$\overline{b}(x,u)[\cdot]\left(f(x)[-]f(u)\right)\geq f^*(u,x\Theta u),\forall x,u\in R^n\quad(2)$$

where

$$\overline{b}(x,u)=\lim_{\lambda\to 0^+}\sup b(x,u,\lambda)=\lim_{\lambda \searrow 0} b(x,u,\lambda).$$

Proof: Since f is $(h,\varphi)-B-$vex in $u\in X$ we have

$$\lambda b(x,u,\lambda)[\cdot]f(x)[+](1-\lambda b(x,u,\lambda))[\cdot]f(u)\geq$$
$$f\left(\lambda\otimes x\oplus(1-\lambda)\otimes u\right)$$

Therefore

$$\lambda b(x,u,\lambda)[\cdot]f(x)[-]\lambda b(x,u,\lambda)[\cdot]f(u)\geq$$
$$f\left(\lambda\otimes x\oplus(1-\lambda)\otimes u\right)[-]f(u)$$

so

$$\lambda b(x,u,\lambda)[\cdot]\left(f(x)[-]f(u)\right)\geq$$
$$f\left(\lambda\otimes x\oplus(1-\lambda)\otimes u\right)[-]f(u)$$

which yield

$$b(x,u,\lambda)[\cdot]\left(f(x)[-]f(u)\right)\geq$$
$$\frac{1}{\lambda}[\cdot]\left(f\left(\lambda\otimes x\oplus(1-\lambda)\otimes u\right)[-]f(u)\right)$$

Going to the limit in the above relation we have



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$$\lim_{\substack{y\to x\\ \lambda\to 0^+}}\sup b(x,u,\lambda)[\cdot]\big(f(x)[-]f(y)\big)\geq$$

$$\lim_{\substack{y\to x\\ \lambda\to 0^+}}\sup \frac{1}{\lambda}[\cdot]\big(f\left(\lambda\otimes x\oplus(1-\lambda)\otimes y\right)[-]f(y)\big)$$

thus

$$\lim_{\substack{y\to x\\ \lambda\to 0^+}}\sup b(x,u,\lambda)[\cdot]\big(f(x)[-]f(y)\big)\geq$$

$$\lim_{\substack{y\to x\\ \lambda\to 0^+}}\sup \frac{1}{\lambda}[\cdot]\big(f\left(y\oplus\lambda\otimes(x\Theta y)\right)[-]f(y)\big)$$

So we get

$$\overline{b}(x,u)[\cdot]\big(f(x)[-]f(u)\big)\geq f^*(u,x\Theta u). \quad \blacksquare$$

**Definition 2.3.** Let $\rho$ be a real number. A locally Lipschitz function $f:X\to R$ is said to be $(h,\varphi)-(\rho,B,\eta)-$invex at $u\in X$ with respect to some functions $\eta,\theta: X\times X\to R^n$, $\big(\theta(x,u)\neq 0 \text{ whenever } x\neq u\big)$,if we have

$$b(x,u)[\cdot]\big(f(x)[-]f(u)\big)\geq$$

$$f^*(u,\eta(x,u))[+]\rho[\cdot]\|\theta(x,u)\|^2_{(h,\varphi)},\forall x\in X \quad (3)$$

$f$ is said to be $(h,\varphi)-(\rho,B,\eta)-$invex on $X$ if it is $(h,\varphi)-(\rho,B,\eta)-$invex at every $u\in X$.

If $\rho>0$, then f is said to be strongly $(h,\varphi)-(\rho,B,\eta)-$invex . If $\rho=0$, then f is said to be $(h,\varphi)-(B,\eta)-$invex . If $\rho>0$, then f is said to be weakly $(h,\varphi)-(B,\eta)-$invex .

**Definition 2.4.** A locally Lipschitz function $f:X\to R$ is said to be $(h,\varphi)-(B,\eta)-$pseudo-invex at $u\in X$, if we have

$$f^*(u,\eta(x,u))\geq 0\Rightarrow$$

$$b(x,u)[\cdot]f(x)\geq b(x,u)[\cdot]f(u),\forall x\in X \quad (4)$$

$f$ is said to be $(h,\varphi)-(B,\eta)-$pseudo-invex on $X$ if it is $(h,\varphi)-(B,\eta)-$pseudo-invex at every $u\in X$.

If $f^*(u,\eta(x,u))\geq 0\Rightarrow$

$$b(x,u)[\cdot]f(x)> b(x,u)[\cdot]f(u),\forall x\in X \quad (5)$$

then we gain the strictly $(h,\varphi)-(B,\eta)-$pseudo-invex definition.

**Definition 2.5.** A locally Lipschitz function $f:X\to R$ is said to be $(h,\varphi)-(B,\eta)-$quasi-invex at $u\in X$, if we have

$$f(x)\leq f(u)\Rightarrow$$

$$b(x,u)[\cdot]f^*(u,\eta(x,u))\leq 0,\forall x\in X \quad (6)$$

$f$ is said to be $(h,\varphi)-(B,\eta)-$quasi-invex on $X$ if it is $(h,\varphi)-(B,\eta)-$quasi-invex at every $u\in X$.

If $f(x)<f(u)\Rightarrow$

$$b(x,u)[\cdot]f^*(u,\eta(x,u))<0,\forall x\in X \quad (7)$$

then we gain the strictly $(h,\varphi)-(B,\eta)-$quasi-invex definition.

## 3. Sufficient Optimality Conditions

Through the rest of this paper, one further assumes that *h* is a continuous one-to-one and onto function with *h*(0) = 0. Similarly, suppose that *φ* is a continuous one-to-one strictly monotone and onto function with *φ*(0) = 0. Under the above assumptions, it is clear that $0[\cdot]\alpha=\alpha[\cdot]0=0$.

Let $f(x),\ g_1(x),g_2(x),...,g_k(x),...$ be locally Lipschtiz functions defined on a non-empty open convex subset $X\subset R^n$. Consider the following semi-infinite programming problems $(P)$:

$$(P)\quad \begin{cases}\min & f(x)\\ s.t. & g_j(x)\leq 0,\quad j=1,2,...\end{cases} \quad (8)$$

Let $D=\{x\in X; g_j(x)\leq 0,\quad j=1,2,...\}$ denote the feasible set of problem $(P)$.

Analogously to the critical point of problem $(P)$ defined in [14], we give a definition as follows.

**Definition 3.1.** The point $x^*\in D$ is said to be a critical point of problem $(P)$, if the set $I=\{i;\ g_i(x^*)=0,\quad \mathrm{i}=1,2,...\}$ is finite (that is $|I|=m<+\infty$), and for each $i\in I$ there exists $l_i\in R_+$ such that

$$0\in\left[\partial^* f(x^*)\oplus\bigoplus_{i\in I} l_i\otimes\partial^* g_i(x^*)\right] \quad (9)$$



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In the remainder of this section, we present the ncessary condition for a critical point $x^*$ to be an global optimal solution of problem $(P)$.

**Theorem 3.1.** Let $x^* \in D$ be a critical point of problem $(P)$. If

(i) $f$ be $B_0-\text{vex}$ and for each $i \in I$, $g_i(x)$ be $B_i-\text{vex}$ at $x^*$. Or

(ii) $f$ be $(B_0,\eta)-\text{invex}$ and for each $i \in I$, $g_i(x)$ be $(B_i,\eta)-\text{invex}$ at $x^*$ with respect to the same $\eta$, and $\bar{b}_0^*(x,x^*) = \lim_{\lambda\to 0^+} b_0(x,x^*,\lambda) > 0$, $\bar{b}_i^*(x,x^*) = \lim_{\lambda\to 0^+} b_i(x,x^*,\lambda) \ge 0$.

Then $x^*$ is an global optimal solution of problem $(P)$.

Proof: (i) Since for each $x \in D$, we have

$$g_i(x) \le 0 = g_i(x^*),\ i \in I \tag{10}$$

Without loss of generality, we suppose $\varphi$ is strictly monotone increasing on $R$, so

$$g_i(x)\left[-\right]g_i(x^*) \le 0,\ i \in I$$

Using $B_i-\text{vexity}$ of $g_i(x)$ at $x^*$ yields

$$g_i^*(x^*,x) \le \bar{b}_i^*(x,x^*)\left[\cdot\right]\left(g_i(x)\left[-\right]g_i(x^*)\right) \le 0,\quad x\in D, i \in I \tag{11}$$

Hence

$$\left[\sum_{i\in I}\right] l_i\left[\cdot\right]g_i^*(x^*,x) \le 0 \tag{12}$$

(9) and Lemma 2.2 yield

$$0 \le \max\left\{\left(\xi^{*T}x\right)_{(h,\varphi)};\ \xi^* \in \left[\partial^* f(x^*) \oplus \bigoplus_{i\in I} l_i \otimes \partial^* g_i(x^*)\right]\right\} =$$
$$= f^*(x^*,x)\left[+\right]\left[\sum_{i\in I}\right] l_i\left[\cdot\right]g_i^*(x^*,x)$$

This along with (12) yield $f^*(x^*,x) \ge 0$.

Using $B_0-\text{vexity}$ of $f$ at $x^*$ yields

$$\bar{b}_0^*(x,x^*)\left[\cdot\right]\left(f(x)\left[-\right]f(x^*)\right) \ge f^*(x^*,x) \ge 0 \tag{13}$$

Thus, from (13), it follows that $f(x) \ge f(x^*),\ \forall x \in D$ which completes the proof of (i).

(ii) It can be proved by following on the lines of(i)■

**Theorem 3.2.** Let $x^* \in D$ be a critical point of problem $(P)$. If one of the following assumption conditions is satisfied

(i) $f$ is $(\rho_0,B,\eta)-\text{invex}$ and for each $i \in I$, $g_i(x)$ be $(\rho_i,B,\eta)-\text{invex}$ at $x^*$ with respect to the same $\eta,\theta$, and $\bar{b}^*(x,x^*) = \lim_{\lambda\to 0^+} b(x,x^*,\lambda) > 0,\ \forall x \in D$ and $\left(\rho_0 + \sum_{i\in I} l_i\rho_i\right) \ge 0$;

(ii) $f$ is strictly $(B_0,\eta)-\text{quasi-invex}$ and for each $i \in I$, $g_i(x)$ be $(B_i,\eta)-\text{quasi-invex}$ at $x^*$ with respect to the same $\eta$, $\bar{b}_0^*(x,x^*) = \lim_{\lambda\to 0^+} b(x,x^*,\lambda) > 0,$, and $\bar{b}_i^*(x,x^*) = \lim_{\lambda\to 0^+} b(x,x^*,\lambda) > 0,\ \forall x \in D$.

Then $x^*$ is an global optimal solution of problem $(P)$.

Proof: Lemma 2.2 and (9) yield

$$f^*(x^*,\eta(x,x^*))\left[+\right]\left[\sum_{i\in I}\right] l_i\left[\cdot\right]g_i^*(x^*,\eta(x,x^*)) \ge 0,\quad \forall x \in D \tag{14}$$

Without loss of generality, we now suppose that $\varphi$ is strictly monotone increasing on $R$

(i) Now, by the $(\rho,B,\eta)-\text{invexity}$ assumptions of $f$ and $g_i$, we have

$$\bar{b}^*(x,x^*)\left[\cdot\right]\left(f(x)\left[-\right]f(x^*)\right) \ge$$
$$f^*(x^*,\eta(x,x^*))\left[+\right]\rho_0\left[\cdot\right]\left\|\theta(x,x^*)\right\|^2_{(h,\varphi)}$$
$$\bar{b}^*(x,x^*)\left[\cdot\right]\left(g_i(x)\left[-\right]g_i(x^*)\right) \ge$$
$$g_i^*(x^*,\eta(x,x^*))\left[+\right]\rho_i\left[\cdot\right]\left\|\theta(x,x^*)\right\|^2_{(h,\varphi)},\ i \in I$$

which yield

$$\bar{b}^*(x,x^*)\left[\cdot\right]\left(f(x)\left[-\right]f(x^*)\left[+\right]\left[\sum_{i\in I}\right] l_i\left[\cdot\right]g_i(x)\right)$$
$$\ge f^*(x^*,\eta(x,x^*))\left[+\right]\left[\sum_{i\in I}\right] l_i\left[\cdot\right]g_i^*(x^*,\eta(x,x^*))\left[+\right]$$
$$\left(\rho_0 + \sum_{i\in I} l_i\rho_i\right)\left[\cdot\right]\left\|\theta(x,x^*)\right\|^2_{(h,\varphi)} \ge 0$$

hence, $f(x)\left[+\right]\left[\sum_{i\in I}\right] l_i\left[\cdot\right]g_i(x) \ge f(x^*)$ (15)



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Since for each $x \in D$, $g_i(x) \le 0$, hence $\left[\sum_{i \in I}\right] l_i [\cdot] g_i(x) \le 0$, (15) yields that

$$f(x) \ge f(x^*),\ \forall x \in D \tag{16}$$

which completes the proof of (i).

(ii) Since for each $x \in D$, we have $g_i(x) \le 0 = g_i(x^*),\ i \in I$, the $(B_i, \eta)-$quasi-invexity of $g_i$ yields that $\bar{b}_i^*(x, x^*)[\cdot] g_i^*(x^*, \eta(x, x^*)) \le 0,\ \forall x \in D, i \in I$.

Hence

$$\left[\sum_{i \in I}\right] l_i [\cdot] g_i^*(x^*, \eta(x, x^*)) \le 0, \forall x \in D \tag{17}$$

(14) and (17) yield

$$f^*(x^*, \eta(x, x^*) \ge 0 \tag{18}$$

If there exists a $\bar{x} \in D$, such that $f(x) < f(x^*)$, then the strictly $(B_0, \eta)-$quasi-invexity yields

$$\bar{b}_0^*(x, x^*)[\cdot] f^*(x^*, \eta(x, x^*)) < 0 \tag{19}$$

which is contradicted with (18), the contradiction yields (16) holds and completes the proof. ■

**Theorem 3.3.** Let $x^* \in D$, $|I| = m < +\infty$, for $i \in I$, there exist $l_i \in R_+$ such that

$$0 \in \left[\partial^* f(x^*) \oplus \partial^* \left(\left[\sum_{i \in I}\right] l_i [\cdot] g_i\right)(x^*)\right] \tag{20}$$

If $f$ is $(B_0, \eta)-$pseudo-invex at $x^*$ and, $\left[\sum_{i \in I}\right] l_i [\cdot] g_i(x)$ is $(\rho, B, \eta)-$invex at $x^*$ with respect to the same $\eta$

$\bar{b}_0^*(x, x^*) = \lim_{\lambda \to 0^+} b(x, x^*, \lambda) > 0,$ and

$\bar{b}^*(x, x^*) = \lim_{\lambda \to 0^+} b(x, x^*, \lambda) \ge 0,\ \forall x \in D$.

Then $x^*$ is an global optimal solution of problem $(P)$.

Proof: Lemma 2.2 and (20) yield

$$f^*(x^*, \eta(x, x^*))[+]\left(\left[\sum_{i \in I}\right] l_i [\cdot] g_i\right)^*(x^*, \eta(x, x^*)) \ge 0, \quad \forall x \in D \tag{21}$$

From the $(\rho, B, \eta)-$invexity of $\left[\sum_{i \in I}\right] l_i [\cdot] g_i(x)$ we have

$$\left(\left[\sum_{i \in I}\right] l_i [\cdot] g_i(x)\right)^*(x^*, \eta(x, x^*))[+]\rho[\cdot]\|\theta(x, x^*)\|^2_{(h,\varphi)} \le \bar{b}^*(x, x^*)[\cdot]\left(\left[\sum_{i \in I}\right] l_i [\cdot] g_i(x)[-]\left[\sum_{i \in I}\right] l_i [\cdot] g_i(x^*)\right) \le 0, \quad \forall x \in D \tag{22}$$

hence,

$$\left(\left[\sum_{i \in I}\right] l_i [\cdot] g_i(x)\right)^*(x^*, \eta(x, x^*)) \le 0,\ \forall x \in D.$$

This with (21) yield

$$f^*(x^*, \eta(x, x^*)) \ge 0, \forall x \in D$$

(23)

Following the $(B_0, \eta)-$pseudo-invexity of $f$ at $x^*$, we complete the proof. ■

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