"Mircea cel Batran" Naval Academy Scientific Bulletin, Volume XVIII – 2015 – Issue 2
Published by "Mircea cel Batran" Naval Academy Press, Constanta, Romania // The journal is indexed in: PROQUEST SciTech Journals, PROQUEST Engineering Journals, PROQUEST Illustrata: Technology, PROQUEST Technology Journals, PROQUEST Military Collection PROQUEST Advanced Technologies & Aerospace

THE TRANSMUTED GENERALIZED PARETO DISTRIBUTION. STATISTICAL INFERENCE AND SIMULATION RESULTS

Romică TRANDAFIR[1]
Vasile PREDA[2]
Sorin DEMETRIU[3]
Ion MIERLUS-MAZILU[4]
[1]Department of Mathematics and Computer Science Technical University of Civil Engineering Bucharest
[2]Faculty of Mathematics and Computer Science University of Bucharest
[3]Department of Structural Mechanics Technical University of Civil Engineering Bucharest
[4]Department of Mathematics and Computer Science Technical University of Civil Engineering Bucharest

Abstract: *In this paper for the Transmuted Generalized Pareto Distribution introduced in [14] we estimate the distribution parameters using the method of moments, maximum likelihood method and Method of probability-weighted moments. For different values of parameters we generate samples of volume 1000 and determine from these samples the mean and standard deviation comparing them to the theoretical. We study the performance of the estimating methods used considering the bias and the root mean squar error and we conclude that are adequate. For the consolidated presentation of the subject approached the paper contains important part of the paper [14].*

Some mathematical properties of the new distribution are presented in this paper.
Mathematics Subject Classification (2010): 62P30, 62N02, 62N05
Key words: *transmuted probability distributions, quadratic rank transmutation map, distribution of order statistics, estimating of parameters, numerical simulation.*

Introduction

In recent years there have been considerable efforts in finding statistical models, not necessarily symmetrical to represent real world phenomena. Given that many of these phenomena are not symmetrical, the efforts were directed towards skewed distributions from other popular distributions symmetrical or not. Asymmetrical patterns that express different degrees of asymmetry are a useful tool in modeling real world phenomena. Starting from a symmetrical distribution with cumulative distribution function $G(x)$ and probability density function $g(x)$, Azzalini [3] proposes asymmetric distribution whose probability density function is $f(x) = 2g(x)G(\beta x)$, where β is the parameter the asymmetry. Shaw and Buckley [12] investigate a novel technique for introducing skewness or kourtosis into symetric or other distribution. Aryal and Tsokos [1] use quadratic rank transmutation map to generate a flexible family of probability distribution starting from extreme value distribution and generalize the two parameter Weibull distribution [2]. Merovci [9], Merovci and Elbatal [8], Elbatan and Elgarhy [4], M Merovci and Puka [10], generalize different kind of Lindley distribution and Pareto distribution using quadratic rank transmutation map obtaining new distributions with applications in reliability. Elbatal et al. [5] consider like base distribution linear exponential distribution and by quadratic rank transmutation map obtain transmuted generalized linear exponential distribution which can use in modeling of life time data. Khan and King [11] introduce transmuted modified Weibull distribution as an important competitive model for life time distributions. The transmuted additive Weibull distribution introduced by Elbatal and Aryal [6] can be used to model lifetime data. The purpose of this paper is to investigate a probability distribution that can be obtained from an asymmetric distribution, namely generalized Pareto distribution and that can be used for modeling and analyzing real-world data.

Quadratic rank transmutation map

Definition 1. [1] *A functional composition of the cumulative distribution function of one probability distribution $F(x)$ with the inverse cumulative distribution function of another $G(x)$,*

$$\mathcal{R}_{GF}(u) = F\left(G^{-1}(u)\right) \qquad (1)$$

is called the transmutation map, where G is considered as the base distribution and F as the modulated distribution.

Obviously, one can also define mutual representation

$$\mathcal{R}_{FG}(u) = G\left(F^{-1}(u)\right) \qquad (2)$$

thus obtaining a pair of rank transmutation maps. Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(y) = \inf_{x \in R}\left\{F(x) \ge y \middle| y \in [0,1]\right\} \qquad (3)$$

The functions $\mathcal{R}_{FG}(u)$ and $\mathcal{R}_{GF}(u)$ both map the unit interval $I = [0,1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $\mathcal{R}_{FG}(0)=0$ and $\mathcal{R}_{GF}(0)=1$.

Definition 2. [1] *A Quadratic Rank Transmutation Map, (QRTM), is defined as*

$$\mathcal{R}_{FG}(u)(u)=u+\lambda u(1-u)\ ,\ |\lambda|\le 1 \tag{4}$$

from which it follows that the cdf's satisfy the relationship

$$F(x)=(1+\lambda)G(x)-\lambda(G(x))^2 \tag{5}$$

which on differentiation yields,

$$f(x)=g(x)[1+\lambda-2\lambda G(x)] \tag{6}$$

where $g(x)$ *and* $f(x)$ *are the corresponding pdfs associated with cdf* $G(x)$ *and* $F(x)$ *respectively.*

The effect of the QRTM is to introduce skew to a symmetric base distribution. There is no specific requirement that the base distribution *G* be symmetric. However, if the *G* distribution is symmetric about the origin, in the sense that *G* (*x*) = G (-*x*), we have the result that the distribution of the square of the transmuted random variable is identical to that of the distribution of the square of the original random variable. A consequence of this is that if the original distribution is symmetric, then the QRTM preserves all even moments [12].

Transmuted generalized Pareto distribution

Definition 3. [13] *Let a random variable X be defined as*

$$X=\frac{b}{a}\left(1-e^{-aY}\right) \tag{7}$$

where a, b are parameters and $Y \sim> Exp(1)$*, a random variable with the standard exponential distribution. If c is the threshold or lower bound of X, then the distribution of X is the 3-parameter generalized Pareto distribution, given by*

$$G(x,a,b,c)=\begin{cases}1-\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}} & \{a<0\ and\ c\le x\}\ or\ \left\{a>0\ and\ c\le x\le c+\frac{b}{a}\right\}\\ 1-e^{-\frac{x-c}{b}} & a=0\ and\ x\ge c\end{cases} \tag{8}$$

where c is a location parameter, b>0 is a scale parameter, a is a shape parameter.

The density probability function of random variable X is given by

$$g(x,a,b,c)=\begin{cases}\frac{1}{b}\left[1-\frac{a}{b}(x-c)\right]^{\frac{1}{a}-1} & \{a<0\ and\ c\le x\}\ or\ \left\{a>0\ and\ c\le x\le c+\frac{b}{a}\right\}\\ \frac{1}{b}e^{-\frac{x-c}{b}}, & a=0\ and\ x\ge c\end{cases} \tag{9}$$

Definition 4. *A random variable X is said to have a transmuted generalized Pareto probability distribution with parameters* $a\in R, b>0$, $c>0$ *and* $|\lambda|\le 1$ *if its cdf is given by*

$$F(x,a,b,c,\lambda)=\begin{cases}\left\{1-\left[1-\frac{a}{b}(x-c)\right]^{\frac{1}{a}}\right\}\left\{1+\lambda\left[1-\frac{a}{b}(x-c)\right]^{\frac{1}{a}}\right\},\\ for\ \{a<0\ and\ c\le x\}\ or\ \left\{a>0\ and\ c\le x\le c+\frac{b}{a}\right\}\\ \left(1-e^{-\frac{x-c}{b}}\right)\left(1+\lambda e^{-\frac{x-c}{b}}\right), & a=0\ and\ x\ge c\end{cases} \tag{10}$$

where λ *is the transmuted parameter.*

The density probability function of random variable X is given by

$$f(x,a,b,c,\lambda)=\begin{cases}\frac{1}{b}\left[1-\frac{a}{b}(x-c)\right]^{\frac{1}{a}-1}\left\{1-\lambda+2\lambda\left[1-\frac{a}{b}(x-c)\right]^{\frac{1}{a}}\right\}, & x>c, a\ne 0\\ \frac{1}{b}e^{-\frac{x-c}{b}}\left(1-\lambda+2\lambda e^{-\frac{x-c}{b}}\right), & x>c, a=0\end{cases} \tag{11}$$

Note that the generalized Pareto distribution is a special case for $\lambda = 0$ of the transmuted generalized Pareto distribution.

Statistic properties

Quantiles

The quantile x_q of transmuted generalized Pareto distribution is the real solution of the equation $F(x,a,b,c,\lambda) = q$. The median can be obtained by setting $q = 0.5$.

$$X_q = c + \frac{b}{a}\left\{1 - \left[\frac{\lambda - 1 + \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right]^a\right\}, \quad for\ a \neq 0 \tag{12}$$

$$X_q = c - b\,ln\left[\frac{\lambda - 1 + \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right], \ for\ a = 0\ , 0 < q < 1 \tag{13}$$

Random number generation

Using the method of inversion we can generate random numbers from a transmuted generalized Pareto probability distribution replacing in equations (12) and (13) *q* by $U \sim> \mathcal{U}(0,1)$.

Moments

The moments of a transmuted generalized Pareto distributed random variable *X* are given by the following proposition.

Proposition 1. *The moments* $E[X^n]$, $n = \overline{1,3}$, *of transmuted generalized Pareto distributed random variable X are given as*

$for\ \ a > 0$

$$E[X] = c + \frac{b}{a+1} - \lambda\frac{b}{(a+1)(a+2)}$$

$$E[X^2] = -\lambda b\frac{3b + 2c + 4ac}{(a+1)(a+2)(2a+1)} + \frac{2a^2c^2 + 4abc + 3ac^2 + 2b^2 + 2bc + c^2}{(a+1)(2a+1)}$$

$$E[X^3] = -3b\lambda\frac{18a^3c^2 + 27a^2bc + 27a^2c^2 + 11ab^2 + 27abc + 13ac^2 + 7b^2 + 6bc + 2c^2}{(a+1)(a+2)(2a+1)(3a+1)(3a+2)} + $$
$$+\frac{6a^3c^3 + 18a^2bc^2 + 11a^2c^3 + 18ab^2c + 15abc^2 + 6ac^3 + 6b^3 + 6b^2c + 3bc^2 + c^3}{(a+1)(2a+1)(3a+1)} \tag{14}$$

$and\ \ for\ \ a = 0$

$$E[X] = b - \frac{\lambda}{2}b + c$$

$$E[X^2] = -\frac{3}{2}\lambda b^2 + 2b^2 + 2bc + c^2 - bc\lambda$$

$$E[X^3] = -\frac{9}{2}b^2c\lambda - \frac{21}{4}b^3\lambda + 3bc^2 + 6b^3 - \frac{3}{2}bc^2\lambda + 6b^2c + c^3$$

Order statistics

In statistics, the *k*th order statistic of a statistical sample is equal to its *k*th smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. We know that if $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ denotes the order statistics of a random sample $X_1, X_{(2)},..., X_{(n)}$ from a continuous population with cdf $F(x)$ and pdf $f(x)$ then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}f(x)[F(x)]^{j-1}[1-F(x)]^{n-j} \tag{15}$$

It is useful in applications the pdf of largest order statistic $X_{(n)}$, given by

$$f_{X_{(n)}}(x)=n\left\{\frac{1}{b}\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}-1}\left[1-\lambda+2\lambda\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\right\}\times$$
$$\times\left\{\left[1-\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\left[1+\lambda\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\right\}^{n-1} \tag{16}$$

for $a\neq 0$ and

$$f_{X_{(n)}}(x)=\frac{n}{b}e^{-\frac{x-c}{b}}\left(1-\lambda+2\lambda e^{-\frac{x-c}{b}}\right)\left(1-e^{-\frac{x-c}{b}}\right)^{n-1}\left(1+\lambda e^{-\frac{x-c}{b}}\right)^{n-1} \tag{15'}$$

for $a=0$.

It is also useful in applications the pdf of smallest order statistic $X_{(1)}$, given by

$$f_{X_{(1)}}(x)=n\left\{1-\left[1-\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\left[1+\lambda\left(1-\frac{a}{b}(x-c)\right)^{\frac{1}{a}}\right]\right\}^{n-1} \tag{17}$$

for $a\neq 0$, and

$$f_{X_{(1)}}(x)=n\frac{1}{b}e^{-\frac{x-c}{b}}\left(1-\lambda+2\lambda e^{-\frac{x-c}{b}}\right)e^{-(n-1)\frac{x-c}{b}}\left(1-\lambda+\lambda e^{-\frac{x-c}{b}}\right)^{n-1} \tag{17'}$$

for $a=0$.

Estimation of parameters

Method of moments (MOM)

In the case of $a>0$, let $X_1,X_2,...,X_n$ be a sample of random variable *X*, to determine the values for vector θ we must solve the system

$$E[X]=\overline{x}\ ,E[X^2]=\overline{x^2}\ ,\ E[X^3]=\overline{x^3}\ , \tag{18}$$

where $E[X^i], i=\overline{1,3}$ are given by Proposition 1 and $\overline{x^i}\ , i=\overline{1,3}$ are empirical moments of order *i*. Using a numerical method we can solve the system (18) considering $c=X_1\ (=min(X))$ since this distribution models the exceeding values of a real phenomenon.

For a sample size 1000 generated by the inversion method with $\theta_0=(a_0,b_0,c_0,\lambda_0)$ the MOM obtained estimators and the corresponding moments are given in the next table:

Parameter vector	Theoretical expectation	Sample mean	Theoretical variance	Sample variance
$\theta_0=(a_0=1.5, b_0=2.5, c_0=10, \lambda_0=0.75)$	10.698	10.78	0.201	0.229
$\theta_0=(\hat{a}=1.16, \hat{b}=1.913, \hat{c}=10.001, \hat{\lambda}=0.38)$	10.78	10.797	0.229	0.239

We consider the parameter $a = 0$. From the system (15) we obtain the estimators of parameters $b^2 = \frac{4S^2}{5-(1+\lambda)^2}$,$c = \bar{x} - \frac{S\cdot(2-\lambda)}{\sqrt{5-(1+\lambda)^2}}$, where $S^2 = \overline{x^2} - \bar{x}^2$, λ being obtained from the last equation of previous system after replacing *b* and *c* depending on λ. Considering sample size 1000 generated by the inversion method with $\theta_0 = (b_0, c_0, \lambda_0)$ the MOM obtained estimators and the corresponding moments are given in the next table:

Parameter vector	Theoretical expectation	Sample mean	Theoretical variance	Sample variance
$\theta_0 = (b_0 = 1.5, c_0 = 10, \lambda_0 = 0.75)$	10.938	10.938	1.079	1.099
$\hat{\theta}_{MOM} = (\hat{b} = 1.494, \hat{c} = 10, \hat{\lambda} = 0.743)$	10.939	10.941	1.094	1.091
$\theta_0 = (b_0 = 2.5, c_0 = 15, \lambda_0 = -0.5)$	18.125	18.126	7.422	7.403
$\hat{\theta}_{MOM} = (\hat{b} = 2.502, \hat{c} = 15.012, \hat{\lambda} = -0.489)$	18.126	18.125	7.407	7.395

To obtain the MOM estimators one can also solve the system

$$E[X] = \bar{x}\ ,\ Var[X] = S^2\ ,\ Skew(X) = \gamma_1 \tag{18'}$$

Where S^2 and γ_1 are empirical variance and skewness respectively, i.e.

$$\begin{cases} b + c - \lambda \dfrac{b}{2} = \bar{x} \\ \dfrac{b^2}{4}\left[5 - (\lambda+1)^2\right] = S^2 \\ \dfrac{2\left(8 - 3\lambda - 3\lambda^2 - \lambda^3\right)}{\sqrt{\left(5 - (\lambda+1)^2\right)^3}} = \gamma_1 \end{cases} \tag{18''}$$

Sometimes it is useful in practical applications kurtosis

$$K(X) = \frac{48 - 24\lambda - 16\lambda^2 - 4\lambda^3 - \lambda^4}{\left[5 - (\lambda+1)^2\right]^2}.$$

Method of probability-weighted moments (PWM)

Consider the r^{th} probability-weighted moments W_r given by

$$Wr = E\left[x(F)(1-F)^r\right] \tag{19}$$

where $r = \overline{0,2}$. We approximate *Wr* by $w_r = \frac{1}{N}\sum_{i=1}^{N-r}\frac{C_{N-i}^r}{C_{N-1}^r}X_{(i)}$ [7], where *N* is the selection volume and $X_{(1)} \le X_{(2)} \le ... \le X_{(N)}$ denotes the order statistics of a random sample $X_1, X_2, ..., X_n$. For $a = 0$ *and* $|\lambda| \le 1$, $\lambda \ne 0$, we obtain the system

$$\begin{cases} b + c - \frac{1}{2}b\lambda = w_0 \\ \frac{b}{4} + \frac{c}{2} - \frac{1}{6}b\lambda + \frac{1}{24}b\lambda^3 = w_2 \\ \frac{b}{9} + \frac{c}{3} - \frac{1}{180}b\lambda^3 + \frac{1}{30}b\lambda^2 - \frac{1}{12}b\lambda = w_2 \end{cases} \quad (19')$$

Solving the system (19') we obtain $\hat{\theta} = (\hat{b}, \hat{c}, \hat{\lambda})$ a estimating vector for $\theta = (b, c, \lambda)$, i.e. $\hat{b} = 12(2w_1 - w_0)\frac{1}{\hat{\lambda}^2 + 2\hat{\lambda} - 6}$, $\hat{c} = w_0 + 6(2w_1 - w_0)\frac{\hat{\lambda} - 2}{\hat{\lambda}^2 + 2\hat{\lambda} - 6}$ and $\hat{\lambda}$ is the solution of equation $\frac{\lambda^3 - 6\lambda^2 - 15\lambda + 40}{\lambda^2 + 2\lambda - 6} + 5\frac{3w_2 - w_0}{2w_1 - w_0} = 0$.

Considering sample size 1000 generated by the inversion method with $\theta_0 = (b_0, c_0, \lambda_0)$ the PWM obtained estimators and the corresponding moments are given in the next table:

Parameter vector	Theoretical expectation	Sample mean	Theoretical variance	Sample variance
$\theta_0 = (b_0 = 8, c_0 = 18, \lambda_0 = 0.5)$	24	24.397	6.633	7.032
$\hat{\theta}_{PWM} = (\hat{b} = 8.603, \hat{c} = 17.519, \hat{\lambda} = 0.401)$	24.397	24.549	7.496	7.422
$\theta_0 = (b_0 = 5, c_0 = 25, \lambda_0 = -0.6)$	31.5	31.625	5.5	5.886
$\hat{\theta}_{PWM} = (\hat{b} = 5.603, \hat{c} = 24.114, \hat{\lambda} = -0.681)$	31.625	32.205	6.2	5.864

Method of maximum likelihood estimation (MLE)

We consider the likelihood function, for $a = 0$

$$L(X;\theta) = \prod_{i=1}^{N} f(X, b, c, \lambda) = \prod_{i=1}^{N} \frac{1}{b} e^{-\frac{X_i - c}{b}} \left(1 - \lambda + 2\lambda e^{-\frac{X_i - c}{b}}\right) \quad (20)$$

and its logarithm

$$\mathcal{L}(X;\theta) = \ln\left(L(X;\theta)\right) \quad (21)$$

To obtain the $\hat{\theta}_{MLE} = (\hat{b}, \hat{c}, \hat{\lambda})$ a estimating MLE vector for $\theta = (b, c, \lambda)$ one must solve the system

$$\frac{\partial \mathcal{L}(X;b,c,\lambda)}{\partial b} = 0, \quad \frac{\partial \mathcal{L}(X;b,c,\lambda)}{\partial c} = 0, \quad \frac{\partial \mathcal{L}(X;b,c,\lambda)}{\partial \lambda} = 0 \quad (22)$$

ie:

$$\begin{cases} N \cdot b + \sum_{i=1}^{N} (c - X_i) \left(1 + \frac{2\lambda e^{-\frac{X_i - c}{b}}}{1 - \lambda + 2\lambda e^{-\frac{X_i - c}{b}}}\right) = 0, \\ N + 2\lambda \sum_{i=1}^{N} \frac{e^{-\frac{X_i - c}{b}}}{1 - \lambda + 2\lambda e^{-\frac{X_i - c}{b}}} = 0, \\ \sum_{i=1}^{N} \frac{2e^{-\frac{X_i - c}{b}} - 1}{1 - \lambda + 2\lambda e^{-\frac{X_i - c}{b}}} = 0 \end{cases} \quad (22')$$

Considering sample size 1000 generated by the inversion method with $\theta_0 = (b_0, c_0, \lambda_0)$ the MLE obtained estimators and the corresponding moments are given in the next table:

Parameter vector	Theoretical expectation	Sample mean	Theoretical variance	Sample variance
$\theta_0 = (b_0 = 2.5, c_0 = 15, \lambda_0 = 0.65)$	16.665	16.645	3.371	3.287
$\hat{\theta}_{MLE} = (\hat{b} = 1.223, \hat{c} = 14.778, \hat{\lambda} = -1)$	16.624	16.592	2.245	2.202
$\theta_0 = (b_0 = 4, c_0 = 30, \lambda_0 = -0.85)$	35.614	36.083	19.187	23.343
$\hat{\theta}_{MLE} = (\hat{b} = 4.193, \hat{c} = 29.779, \hat{\lambda} = -0.999)$	35.946	36.078	23.707	22.352

Like measure of performances of the parametes obtained above we consider two performance indices, namely standard bias, $bias = \frac{E[\hat{\theta}] - \theta}{\theta}$ and root mean squar error, $RMSE = \frac{\sqrt{E[(\hat{\theta} - \theta)^2]}}{\theta}$, where $E[\hat{\theta}] = \frac{\sum_{i=1}^{n} \hat{\theta}_i}{n}$ and *n* is number of runs.

For $n = 35$, $N = 1000$ the values of performance indices are given in the next tables:

- Case of $a > 0$, $c = X_1$ $(= min(X))=10$

Method	$\theta_0 = (a_0, b_0, \lambda_0)$			$E[\hat{\theta}]$			bias			RMSE		
	a_0	b_0	λ_0	$E[\hat{a}]$	$E[\hat{b}]$	$E[\hat{\lambda}]$	a	b	λ	a	b	λ
MOM	1.5	2.5	0.75	1.104	1.886	-0.267	-0.264	-0.246	-0.568	0.385	0.48	0.611

- Case of $a = 0$

Method	$\theta_0 = (b_0, c_0, \lambda_0)$			$E[\hat{\theta}]$			bias			RMSE		
	b_0	c_0	λ_0	$E[\hat{b}]$	$E[\hat{c}]$	$E[\hat{\lambda}]$	b	c	λ	b	c	λ
MOM	1.5	10	0.75	1.43	9.999	0.462	-0.171	-0.00012	-0.384	0.207	0.0023	0.4505
PWM	8	18	0.5	13.265	19.363	0.909	0.658	0.076	0.819	0.698	0.116	0.833
MLE	4	30	-0.85	3.583	30.3	-1	-0.104	0.01	0.176	0.109	0.012	-0.175

The results from the previous tables regarding the bias and RMSE of parameter estimators entitle us to believe that the proposed methods for estimating these parameters are suitable.

CONCLUSIONS

We have determined the distribution parameters of Transmuted Generalized Pareto Distribution using the method of moments, method of probability-weighted moments and Method of maximum likelihood of the simulated values. We studied performance considering two indicators namely bias and root mean square standard error for different values of parameters and we concluded that the proposed methods for estimating these parameters are suitable.

BIBLIOGRAPHY:

[1] Aryal, R. G., Tsokos, C. P.: On the Transmuted Extreme Value Distribution with Application, *Nonlinear Analysis*, **71** (2009), 1401-1407.

[2] Aryal, R. G., Tsokos, C. P.: Transmuted Weibull Distribution: A Generalized of the Weibull Probability Distribution, *European Journal of Pure and Applied Mathematics,* **4** (2, (2011), 89-102.

[3] Azzalini, A.: References on the skew-normal distribution and related ones, On-line at http://tango.stat.unipd.it/SN/list-publ.pdf.

[4] Elbatan, I. & Elgarhy, M.: Transmuted Quasi Lindley Distribution : A Generalization of the Quasi Lindley Distribution, *International Journal of Pure and Applied Sciences and Technology*, **18** (2) (2013), 59-70

[5] Elbatan, I., Diab, L. S. & Abdul Alim, N. A.: Transmuted Generalized Linear Exponential Distribution of the Quasi Lindley Distribution, *International Journal of Computer Application* (075-8887), **83** (17) (2013), 29-37

[6] Elbatal, I. & Aryal, G: On the Transmuted Additive Weibull Distribution, *Austrian Journal of Statistics,* 42 (2) (2013), 117-132*.*

[7] Maciunas Landwher, J. Matalas, N. C.& Wallis, J. R.: Probability Weighted Moments Compared with Some Traditional Techniques in Estimating Gumbel Parameters and Quantiles, *Wat. Resour. Res*., **15**(5), (1979), 1055-1064.

[8] Merovici, F. & Elbatal, I.: Transmuted Lindley-Geometric Distribution and its Applications, *arXiv:1309.3774v2[stat.ME] 27 Oct 2013*
[9] Merovci, F.: Transmuted Rayleigh Distribution, *Austrian Journal of Statistics,* **42** (1) (2013), 21-31*.*
[10] Merovci, F.and Puka, L.: Transmuted Pareto Distribution, in *ProbStat Forum*, **07** (2014), 1-11.
[11] Khan, M. S. & King, R.: Transmuted Modified Weibull Distribution: A Generalization of the Modified Weibull Probability Distribution, *European Journal of Pure and Applied Mathematics*, **6** (1) (2013), 66-88.
[12] Shaw, W. T. and Buckley, I. R.: The alchemy of probability distributions: beyond Ghram-Charlier expansions and skew-kurtotic-normal distribution from a rank transmutation map, *arXiv preprint arXiv: 0901.0434 (2009).*
[13] Sing, V. P. & Guo, H.: Parameter Estimation for 3-Parameter Generalized Pareto Distribution by the Principle of Maximum Entropy (POME), *Hydrological Sciences Journal*, **40** (2), (1995) 165-181.
[14] Trandafir, R. & Preda, V.: Transmuted Generalized Pareto Distribution, *The 13 th Workshop of Scientific Communications, Department of Mathematics and Computer Science*, Matrix Rom, ISSN 2067-3132, May 23, 2015, 177-182.