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# RISK NEUTRAL DENSITIES AND STATISTICAL HETEROGENEITY

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***Abstract**: Statistical Physics and Information Theory commonly use Shannon's entropy which measures the randomness of probability laws, whereas Economics and the Social Sciences commonly use Gini's index which measures the evenness of probability laws. The problem of shifting from the "principal of maximum entropy" to the more general "principal of maximum heterogeneity", and explore the maximization of statistical heterogeneity was studied by Eliazar and Sokolov [2010].We propose the framework of entropy pricing theory in this regard, introduced by Gulko [1996]. We consider various entropy maximization problems to obtain the risk neutral densities based on Eliazar and Sokolov methodology.*

***Keywords:** Information Theory, Gini's Index, Shannon Entropy, Risk Neutral Densities.*

## INTRODUCTION

The entropic reasoning allows us to approach two different ways of quantifying the statistical heterogeneity of risk-neutral price distribution. The Gin's index is a measure of statistical dispersion and measures the evenness of probability laws where entropy measures the randomness.

The application of entropy in finance can be regarded as the extension of both information entropy and probability entropy. Since last two decades, it has become a very important tool for the methods of portfolio selection and asset pricing. The famous Black-Scholes model [1] assumes the condition of no arbitrage which implies the universe of risk-neutral probabilities. The uniqueness of these risk-neutral probabilities is very crucial. The stock price process is controlled by Geometric Brownian Motion (GBM) in Black and Scholes model and in this framework stochastic calculus is vital. The Entropy Pricing Theory (EPT) was introduced by Les Gulko as an alternative method for the construction of risk-neutral probabilities without relying on stochastic calculus [9,10]. Recently Preda & Sheraz have introduced new approach to obtain the risk neutral densities [16].

The Principle of Maximum Entropy (MEP) has been extensively used to estimate the distribution of an asset from a set of option prices. The problem of shifting from the "principal of maximum entropy" to the more general "principal of maximum heterogeneity", and explore the maximization of statistical heterogeneity was studied by Eliazar & Sokolov [6]. The maximum entropy principle was used to retrieve the risk-neutral density of future stock risks or other asset risks [19]. The Renyi entropy [17] generalizes the frequently used Shannon entropy [18] and it has been used for option price calibration [5]. Recently Preda et al. used Tsallis and Kaniadakis entropy measures for the case of semi-Markov regime switching interest rate models. Preda et al have also introduced the new classes of Lorenz curves by maximizing Tsallis entropy under mean and Gini's equality and inequality constraints [14,15]. For maximum entropy distribution of asset returns, application of entropy maximization problems, and others can be found in [3, 4,13].

Two complimentary problems are discussed in this article: entropy maximization for specified Gini's index value and Gini's maximization for specified Shannon entropy. In Section 2 we present the problem formulation and preliminaries. We present our main results for risk-neutral densities based on Eliazar & Sokolov problems in the framework of entropy pricing theory of Les Gulko. Section 4 concludes our results and future directions.

## PRELIMINARIES

In this section we use the concept of EPT [9, 10] The term market belief is vital in option pricing and the current price of any risky asset indicates this belief. The future picture of the market up (down) reflects a state of maximum possible uncertainty; therefore market belief for the future performance of an efficient price is characterized by maximum uncertainty. Consider a risky asset on time interval $[0,\mathrm{T}]$. Let $Y_T$ be asset price process of $S_T$ at future time $T$, $G$ as state space, a subset of real line $R$, $\mathrm{g}(\mathrm{S}_T)$ the probability densities on $\mathrm{P}$, $f(S_T)$ efficient market belief and $H(g)$ the index of market uncertainty about $Y_T$. The $H(g)$ is defined on the set of beliefs $g(S_T)$ therefore the efficient market belief $f(S_T)$ maximizes $H(g)$.

We can determine $f(S_T)$ given $H(g)$ with some relevant information about current price of $S$. The index of the market uncertainty about $Y_T$ as a Shannon entropy[18], can be written as:

$$H(g) = -E^{g}\left[\ln\ \mathrm{g}\left(Y_T\right)\right] \tag{1}$$

In the above equation $H(g)$ is a functional defined on $g(S_T)$, $f(S_T)$ which maximize $H(g)$ is called the entropy of random variable $Y_T$ and used to measure the degree of uncertainty of $g(S_T)$. The maximum entropy characterizes the market beliefs regardless of the subjective risk preferences and it is useful to find the risk neutral beliefs in incomplete arbitrage free markets. The maximum entropy market belief $f(S_T)$ as a solution to the maximum entropy problem can be written as follows:

$$f = \arg\ \max \left\{ H(g), g \in G \right\} \quad (2)$$

Eliazar and Sokolov [6] have studied the problem of shifting Shannon's entropy to Gini.s index which is called statistical heterogeneity. There are different approaches that estimate statistical heterogeneity such as concepts of dispersion, entropy and equality i.e. measuring the evenness of the probability laws. The approach of equality or measure of evenness is called Gini's index in the field of economics and social sciences. Mathematically Gini's index is given by:

$$G(f) = 1 - \frac{1}{\mu}\int_0^{\infty}\left(\int_x^{\infty} f(z)dz\right)^2 dx \quad (3)$$

where $\mu$ is the mean and in our frame work we replace it by $\frac{S}{P}$. We present our results of risk neutral densities using Eliazar & Sokolov [6] approach in another frame work.

The Gini's distance of a probability law controlled by the probability density function $g(S_T)$, where $-\infty < S_T < \infty$ is given by:

$$D(g) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|S_{T_1} - S_{T_2}\right| g\left(S_{T_1}\right) g\left(S_{T_2}\right) dS_{T_1} dS_{T_2} \quad (4)$$

where $D(g)$ is the average distance between two independent random variables whose probability law is controlled by the probability density function $g(S_T)$ where $-\infty < S_T < \infty$. Gini.s distance is positive and in the case of probability laws supported on the positive half-line the connection between Gini.s index $G(g)$ and Gini.s distance $D(g)$ is:

$$G(g) = \frac{1}{D(g)} \quad (5)$$

The first variation of the functional $D(g)$ is given by [6].

$$\Delta\left[D(g)\right](\chi) = \frac{1}{2}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{S_T} G^*(u)du + \int_{S_T}^{\infty} G_*(u)du\right]\chi(S_T)dS_T$$

(6)

where $\chi(S_T)$ is an arbitrary test function and $-\infty < S_T < \infty$ therefore:

$$G^*(u) = \int_{-\infty}^{u} g\left(u'\right)du', \ -\infty < u < \infty$$

$$G_*(u) = \int_{u}^{\infty} g\left(u'\right)du', \quad -\infty < u < \infty \quad (7)$$

RISK NEUTRAL DENSITIES

We consider the Shannon's entropy maximization problem as the first case. We suppose that all expectations are also well defined and underlying optimization problems admit solutions for some continuous cases. We discuss the law introduced in [6]. This law maximizes Shannon's entropy within the class of probability laws supported on the real line $-\infty < S_T < \infty$ with mean $\mu = \frac{S}{P}$ and a given dispersion, where $S$ and $P$ belong to the prior information set which indicate risky asset and riskless bond price respectively. $E^g$ denotes expectation relative to risk neutral density $g(S_T)$ and $S_T$ is the asset price at time $T$.

THEOREM 3.1 The risk neutral density $g(S_T)$ which solves the Shannon entropy maximization problem subject to the given constraints:

$$\underset{g}{Max}\ -E^g\left[\ln g(Y_T)\right]$$

Subject to

$$E^g\left[I_{\{Y_T\}}\right] = 1 \qquad \text{C-1}$$

$$E^g\left[Y_T\right] = \frac{S}{P} \qquad \text{C-2}$$

$$E^g\left[\left|Y_T - \frac{S}{P}\right|^r\right] = \delta, r > 0 \quad \text{C-3}$$

Then the unique solution of the risk neutral density is given by:

$$g(S_T) = \exp\left[-1 - \lambda_1 - \lambda_2 S_T - \lambda_3\left|S_T - \frac{S}{P}\right|^r\right]$$

where $\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers and determined by using the given constraints C-1, C-2 and C-3.

Proof: We can proof the above result by using the calculus of variations for optimization of functionals (see Luenberger -1969, Borwein - 2003, Eliazar &Sokolov-2010).

Therefore the Lagrangian $L(g,\lambda)$ can be written as:

$$L(g,\lambda) = E^g\left[\ln g(Y_T)\right] + \lambda_1\left(E^g\left[I_{\{Y_T\}}\right] - 1\right) +$$
$$+\lambda_2\left(E^g[Y_T] - \frac{S}{P}\right) + \lambda_3\left(E^g\left[\left|Y_T - \frac{S}{P}\right|^r\right] - \delta\right)$$

where $\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers and we write the first variation $A(g)$ of $L(g,\lambda)$:

$$A(g) = \int_a^b \eta(S_T) g(S_T)\, dS_T$$

The first variation $A(g)$ is given by:

$$\Delta\left[A(g)\right](\chi) = \int_a^b \eta(S_T)\chi(S_T)\, dS_T$$

where $\chi(S_T)$ is a arbitrary test function and $a < S_T < b$. Let $(a,b)$ an interval $-\infty \le a, \infty \ge b$ and consider the convex functional:

$$H(g) = \int_a^b \ln\left(g(S_T)\right) g(S_T)\, dS_T$$

where $g(S_T)$ is a probability density function then first variation of $H(g)$:

$$\Delta\left[H(g)\right](\chi) = \int_a^b \left[1 + \ln g(S_T)\right]\chi(S_T)\, dS_T$$

Now using $\Delta\left[A(g)\right](\chi)$ and $\Delta\left[H(g)\right](\chi)$, we can write the fist variation of $L(g,\lambda)$:

$$\Delta\left[L(g,\lambda)\right](\chi) = \int_{-\infty}^{+\infty} [1 + \ln\left(g(S_T)\right) + \lambda_1 + +\lambda_2 S_T +$$
$$+\lambda_3\left|S_T - \frac{S}{P}\right|^r ]\chi(S_T)\, dS_T$$

Now equating this first variation equal to zero we get:

$$1 + \ln\left(g(S_T)\right) + \lambda_1 + \lambda_2 S_T + \lambda_3\left|S_T - \frac{S}{P}\right|^r = 0$$

$$g(S_T) = \exp\left[-1 - \lambda_1 - \lambda_2 S_T - \lambda_3\left|S_T - \frac{S}{P}\right|^r\right]$$

In the next theorem 3.2 we extend the previous result of theorem 3.1 for the case of weighted entropy. The weighted entropy was first defined by Guiasu [8], considering the two basic concepts of objective probability and subjective utility.

THEOREM 3.2. The risk neutral density $f(S_T)$ which solves the Weighted-Shannon entropy maximization problem:

$$\max_g -E^g\left[u(Y_T)\ln g(Y_T)\right]$$

Subject to the given constraints C-1,C-2,C-3 and $u > 0$. Then the unique solution is given by:

$$f(S_T) = \frac{\exp\left[-1 - \lambda_1 - \lambda_2 S_T - \lambda_3\left|S_T - \frac{S}{P}\right|^r\right]}{u(S_T)}$$

Proof: We can proof the result by using the calculus of variations for optimization of functionals see Luenberger). Now consider the function:

$$F(f, S_T) = -u(S_T)\, f \ln f + \lambda_1 + \lambda_2 S_T + \lambda_3\left|S_T - \frac{S}{P}\right|^r$$

where $\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers. The necessary condition for an extremum is the Euler-Lagrange equation given by:

$$\frac{\partial F}{\partial f} + \frac{\partial}{\partial S_T}\left(\frac{\partial F}{\partial f'}\right) = 0$$

Since $F(f, S_T)$ is independent of $f'$, the derivative of $f(S_T)$ ad Euler-Lagrange equation reduces to:

$$-u(S_T) - u(S_T)\ln f + \lambda_1 + \lambda_2 S_T + \lambda_3\left|S_T - \frac{S}{P}\right|^r = 0$$

Thus we obtain the required result by using the above equation and the proof is complete.

In the next theorem we consider the Gini's maximization problem and replacing randomness by evenness as the underlying measure of heterogeneity.

THEOREM 3.3. Consider the following entropy maximization problem:

$$G(S_T) = E^g\left[I_{\{Y_T > S_T\}}\right],\ \overline{G}(S_T) = 1 - G(S_T)$$

which is equivalent to the convex optimization problem:

$$\min \int_0^\infty \overline{G}(Y_T)^2\, dY_T$$

Subject to C-1,C-2 and C-3 as defined in the theorem 3.1 then the solution of risk neutral density is given by:

$$g(S_T) = \frac{\lambda_3}{2} r(r-1)\left|S_T - \frac{S}{P}\right|^r$$

Proof: The corresponding Lagrangian can be written by using calculus of variations for optimization of functionals (see Luenberger-1969, Eliazar & Sokolov-2010).

$$L(g,\lambda) = \int_0^{\infty} \overline{G}(S_T)^2 \, dY_T + \lambda_1\left(E^g\left[I_{\{Y_T\}}\right] - 1\right) +$$

$$+\lambda_2\left(E^g[Y_T] - \frac{S}{P}\right) + \lambda_3\left(E^g\left[\left|Y_T - \frac{S}{P}\right|^r\right] - \delta\right)$$

The above problem is very similar to the theorem 3.1 ,where $\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers and we write the first variation of $L(g,\lambda)$:

$$\Delta\left[L(g,\lambda)\right](\chi) = \int_0^{+\infty}\left[2\int_0^{S_T} \overline{G}(S_T^{/}) dS_T^{/} + \lambda_1 + \lambda_2 S_T + \lambda_3\left|S_T - \frac{S}{P}\right|^r\right]. \chi(S_T)\, dS_T$$

Thus we get:

$$2\int_0^{S_T} \overline{G}(S_T^{/}) dS_T^{/} + \lambda_1 + \lambda_2 S_T + \lambda_3\left|S_T - \frac{S}{P}\right|^r = 0$$

Differentiating twice both sides of the above equation we get the solution and the proof is complete.

Now we consider the Entropy-Gini-Maximization problem by following the law that maximizes Gini's index within the class of probability laws on the positive half line of $S_T \geq 0$ with mean $\frac{S}{P}$ and given the Gini index.

THEOREM 3.4. Consider the following Entropy-Gini-Maximization problem:

$$\max_g \; -E^g\left[\ln g(Y_T)\right]$$

Subject to the constraints C-1,C-2 and a new constraint C-4:

$$\int_0^{\infty} G\left(S_T^2\right) dS_T = \gamma \qquad \text{C-4}$$

Then $G$ satisfies the differential equation given by:

$$G^{/}(S_T) = \text{c-}\lambda_2 \mathrm{G}(S_T) - \lambda_3 \mathrm{G}(S_T)^2$$

where c is constant of integration and the solution of differential equation is given by:

$$\mathrm{G}(S_T) = \frac{1}{c_3 \exp(-c_1 S_T) + (1 - c_3)}$$

Where $c_1$ and $c_3$ are positive real valued parameters and

$$c_1 = \frac{\psi(c_3)}{\frac{S}{P}}, \psi(c_3) = \frac{\ln(c_3)}{c_3 - 1}, \mathrm{c}_1 = -\lambda_1, \mathrm{c}_3 = -\lambda_3$$

Proof: Following the method of theorem 3.1 we can write the Lagrangian:

$$L(g,\lambda) = E^g\left[\log g(Y_T)\right] + \lambda_1\left(E^g\left[I_{\{Y_T\}}\right] - 1\right) +$$

$$+\lambda_2\left(E^g[Y_T] - \frac{S}{P}\right) + \lambda_3\left(\int_0^{\infty} G(S_T)^2 \, dS_T - \gamma\right)$$

The above problem is very similar to the theorem 3.1 where $\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers and we write the first variation of $L(g,\lambda)$:

$$\Delta\left[L(g,\lambda)\right](\chi) = \int_0^{\infty}\left[1 + \ln(g(S_T)) + \lambda_1 + \lambda_2 S_T + 2\lambda_3 \int_0^{S_T} G(u) du\right]. \chi(S_T)\, dS_T$$

Therefore we get:

$$1 + \ln(g(S_T)) + \lambda_1 + \lambda_2 S_T + 2\lambda_3 \int_0^{S_T} G(u) du = 0$$

Differentiating the above equation we get:

$$\frac{g^{/}(S_T)}{g(S_T)} + \lambda_2 + 2\lambda_3 G(S_T) = 0$$

$$g^{/}(S_T) + \lambda_2 g(S_T) + 2\lambda_3 G(S_T) g(S_T) = 0$$

Since,

$$-G^{//}(S_T) - \lambda_2 G(S_T) - \lambda_3 G(S_T)^2 = 0$$

Therefore we get:

$$G^{/}(S_T) + \lambda_2 G(S_T) + \lambda_3 G(S_T)^2 = c$$

(i)

where c is a constant of integration. The solution of equation can be found using Riccati differential equation see [5] for more details.

We can write equation (i) for its solution:

$$G^{/}(S_T) = c_0 + c_1 G(S_T) + c_2 G(S_T)^2$$

where $c_0, c_1, c_2$ are constants and defined w.r.t

Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3$. We have boundary conditions $G(0)=1$ and $\lim_{S_T \to 0} G(S_T)=0$, Since $G(S_T)$ is a survival probability function then coefficient $c_0=\lambda_0=0$ and the given equation reduces to the Bernoulli form:

$$G^{/}(S_T)=c_1 G(S_T)+c_2 G(S_T)^2$$

i.e.

$$\frac{G^{/}(S_T)}{G(S_T)^2}-c_1 G(S_T)^{-1}=c_2$$

Let us suppose:

$$G(S_T)^{-1}=y \Rightarrow \frac{dy}{dS_T}=-\frac{1}{G^2}\frac{dG(S_T)}{dS_T}$$

i.e.,

$$\frac{dy}{dS_T}+c_1 y=-c_2$$

We get the solution of the above differential equation:

$$y=\frac{c_2}{c_1}+\frac{c_3}{e^{c_1 S_T}}$$

$$G(S_T)^{-1}=\frac{c_2}{c_1}+\frac{c_3}{e^{c_1 S_T}}$$

$$G(S_T)=\frac{1}{\frac{c_2}{c_1}+\frac{c_3}{e^{c_1 S_T}}}$$

Using $G(0)=1$, we get $c_3=1-\frac{c_2}{c_1} \Rightarrow 1-c_3=\frac{c_2}{c_1}$

Therefore we can write:

$$G(S_T)=\frac{1}{c_3\exp(-c_1 S_T)+(1-c_3)}$$

The corresponding mean is given by:

$$\mu=\frac{S}{P}=\int_0^\infty G(S_T)dS_T=\frac{\ln c_3}{c_3-1}.\frac{1}{p},$$

Hence, $\psi(c_3)=\frac{\ln c_3}{c_3-1}$

and we get the value of $c_1=\frac{\psi(c_3)}{\frac{S}{P}}$

THEOREM 3.5. Consider the following Entropy—Gini-Maximization problem:

$$\max_g \int_0^\infty G(S_T^2)dS_T$$

Subject to the constraints C-1,C-2 and a new constraint, C-5, i.e. the Shannon entropy:

$$E^g\left[\ln g(Y_T)\right]=\xi \qquad \text{C-5}$$

Then solution satisfies the differential equation given by:

$$G^{/}(S_T)=\frac{c}{\lambda_3}-\frac{\lambda_2}{\lambda_3}\mathrm{G}(S_T)-\frac{1}{\lambda_3}\mathrm{G}(S_T)^2$$

Proof: Following the method of theorem 3.1 we can write the Lagrangian:

$$L(g,\lambda)=\int_0^\infty G(S_T)^2 dS_T+\lambda_1\left(E^g\left[I_{\{Y_T\}}\right]-1\right)+$$
$$+\lambda_2\left(E^g\left[Y_T\right]-\frac{S}{P}\right)+\lambda_3\left(E^g\left[\log g(Y_T)\right]-\xi\right)$$

$\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers and we write the first variation of $L(g,\lambda)$:

$$\Delta\left[L(g,\lambda)\right](\chi)=\int_0^\infty\left[2\int_0^{S_T}G(u)du+\lambda_1+\lambda_2 S_T+\lambda_3\left(1++\ln\left(g(S_T)\right)\right)\right].$$
$$\chi(S_T)dS_T$$

Therefore we get:

$$2\int_0^{S_T}G(u)du+\lambda_1+\lambda_2 S_T+\lambda_3\left(1++\ln\left(g(S_T)\right)\right)=0$$

Differentiating the above equation we get:

$$\lambda_3 g^{/}(S_T)+\lambda_2 g(S_T)+2G(S_T)g(S_T)=0$$

which is equivalent to ,

$$-\lambda_3 G^{//}(S_T)-\lambda_2 G^{/}(S_T)-\lambda_3\left(G(S_T)^2\right)^{/}=0$$

Therefore we get:

$$\lambda_3 G^{/}(S_T)+\lambda_2 G(S_T)+\lambda_3 G(S_T)^2=c$$

where c is the real valued constant and therefore we have the differential equation given in the statement of the theorem and the proof is complete.

Now we present our frame work to discuss the law that maximizes the Gini's distance within the class of probability laws supported on the real line $-\infty < S_T < +\infty$ with mean $\frac{S}{P}$ and given dispersion.

THEOREM 3.6. Consider the following optimization problem:

$$\max\ D(g)$$

Subject to the constraints C-1,C-2 and C-3 then the solution of risk neutral density is given by:

$$g(S_T) = -r(r-1)\lambda_3\left|S_T - \frac{S}{P}\right|^{r-2},\ \ r > 2$$

Proof: We can proof the above results by following the theorem 3.1. Therefore the Lagrangian $L(g,\lambda)$ can be written as:

$$L(g,\lambda) = D(g) + \lambda_1\left(E^g\left[I_{\{Y_T\}}\right]-1\right) +$$
$$+\lambda_2\left(E^g\left[Y_T\right]-\frac{S}{P}\right)+\lambda_3\left(E^g\left[\left|Y_T-\frac{S}{P}\right|^r\right]-\delta\right)$$

$\lambda_1,\lambda_2,\lambda_3$ are Lagrange multipliers and we write the first variation of $L(g,\lambda)$:

$$\Delta\left[L(g,\lambda)\right](\chi) = \int_{-\infty}^{+\infty}[\frac{1}{2}\int_{-\infty}^{S_T}G^*(u)\,du + \int_{S_T}^{\infty}G_*(u)\,du +$$
$$+\lambda_1+\lambda_2 S_T+\lambda_3\left|S_T-\frac{S}{P}\right|^r]\chi(S_T)\,dS_T$$

Now equating this first variation equal to zero we get:

$$\frac{1}{2}\int_{-\infty}^{S_T}G^*(u)\,du + \int_{S_T}^{\infty}G_*(u)\,du + \lambda_1 + \lambda_2 S_T +$$
$$+\lambda_3\left|S_T-\frac{S}{P}\right|^r = 0$$

Now differentiating the above equation twice we get the required solution and the proof is complete

We discuss another problem which maximizes the Gini's index given the Shannon entropy with the class of probability laws supported on the real line $-\infty < S_T < +\infty$ with mean $\frac{S}{P}$.

THEOREM 3.7. Consider the following optimization problem:

$$\max\ D(g)$$

Subject to the constraints C-1,C-2 and C-5 then $G_*$ is the solution of the differential equation given by:

$$y_*(S_T) = \frac{c}{2\lambda_3} - \frac{1+2\lambda_2}{2\lambda_3}y(S_T) + \frac{1}{2\lambda_3}y^2(S_T)$$

Proof: The corresponding Lagrangian can be written as:

$$L(g,\lambda) = D(g) + \lambda_1\left(E^g\left[I_{\{Y_T\}}\right]-1\right) +$$
$$+\lambda_2\left(E^g\left[Y_T\right]-\frac{S}{P}\right)+\lambda_3\left(E^g\left[\ln g(Y_T)\right]-\xi\right)$$

$\lambda_1,\lambda_2,\lambda_3$ are Lagrange multipliers and we write the first variation of $L(g,\lambda)$:

$$\Delta\left[L(g,\lambda)\right](\chi) = \int_{-\infty}^{+\infty}[\frac{1}{2}\int_{-\infty}^{S_T}G^*(u)du + \int_{S_T}^{\infty}G_*(u)du + \lambda_1 +$$
$$+\lambda_2 S_T + \lambda_3\left(1++\ln\left(g(S_T)\right)\right)]\chi(S_T)\,dS_T$$

Therefore we get:

$$\frac{1}{2}\int_{-\infty}^{S_T}G^*(u)du + \int_{S_T}^{\infty}G_*(u)du + \lambda_1 + + \lambda_2 S_T +$$
$$+\lambda_3\left(1++\ln\left(g(S_T)\right)\right) = 0$$

Differentiating the above equation we get:

$$2\lambda_3 g'(S_T) + 2\lambda_2 g(S_T) + \left(1-2G_*(S_T)\right)g(S_T) = 0$$

This equation is equivalent to

$$-2\lambda_3 G''(S_T) - (1+2\lambda_2)G_*'(S_T) + \left(G_*(S_T)^2\right)'$$

Now we get:

$$-2G'(S_T) - (1+2\lambda_2)G(S_T) + G(S_T)^2 = -c$$

where c is the real valued integration constant and we have the following equation:

$$y' = \frac{c}{2\lambda_3} - \frac{1+2\lambda_2}{2\lambda_3}y + \frac{1}{2\lambda_3}y^2$$

We can solve the above equation using the method described in theorem 3.4.

## CONCLUSIONS

We have studied some entropy maximization problems of Eliazar & Sokolov [6]. We have introduced the frame work of entropy pricing theory in this regard to find out risk neutral densities. The importance of risk neutral densities in modern financial literature is vital which provide a comprehensive package without specifying any model based on stochastic calculus.

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## BIBLIOGRAPHY:

[1] F. Black, M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy, vol 81, pp. 637-651,1973.
[2] J. Borwein, R. Choksi and P. Maréchal, Probability distributions of assets inferred from option prices via the principle of maximum entropy. Journal of Society for Industrial and Applied Mathematics,vol 3, pp. 464-478, 2003.
[3] P.W. Buchen, M. Kelly, The maximum entropy distribution of an asset inferred from option prices, Journal of Finance and Quantitative Analysis,vol 31, pp. 143-159, 1996.
[4] T.M. Cover, J.A., Thomas, Elements of information theory 1st ed, Wiley, New York, 1991.
[5] C.B. Droje, R.C.B. Ian,and C.C. Irene, Option price calibration from Renyi entropy, Physics Letters A, vol 366, pp. 298-307, 2007.
[6] I. Eliazar, I.M. Sokolov, Maximization of statistical heterogeneity: From Shannon's entropy to Gini's index, Physica A, vol 389, pp. 3023-3038, 2010.
[7] S. Dedu., R. Ciumara., Restricted optimal retention in Stop-Loss reinsurance under VaR and CTE risk measures, Proceedings of the Romanian Academy **,** vol (3), pp. 213-217, 2010.
[8] S. Guiasu, Weighted entropy, Reports on Mathematical Physics vol 2(3), pp. 165-179, 1971.
[9] L. Gulko, Dart Boards and Asset Prices: Introducing the Entropy Pricing Theory, Chapter 14 in Advances in Econometrics. T.B. Fomby and R.C, Hill eds, JAI press, Greenwich, CT, 1997.
[10] L. Gulko, The Entropy Theory of Bond Option Pricing, Working Paper, Yale School of Management, October, New Haven, CT, 1995.
[11] W.Y.Guo, Maximum entropy in option pricing: a convex-spline smoothing method. Journal of Futures Markets **21**, pp. 819-832, 2001.
[12] D.G. Luenberger, Optimization by vector space methods, Wiley, New York, 1969.
[13] V. Preda, C. Balcau, Entropy optimization with applications, The Publishing House of Romanian Academy, 2010.
[14] V.Preda, S.Dedu, G.Carmen, New class of Lorenez curves by maximizing Tsallis entropy under mean and Gini equality and inequality constraints, Physica A, vol 436, pp. 925-932, 2015.
[15] V.Preda. S. Dedu., M. Sheraz. ., New measure selection for Hunt-Devolder semi-Markov regime switching interest rate models, Physica A, vol 407, pp. 350-359, 2014.
[16] V. Preda, M. Sheraz, Risk neutral densities in entropy pricing theory of stock options using Lambert function and a new approach, Proceedings of the Romanian Academy, VOL 16(1), pp. 20-27, 2015.
[17] A.Reyni, On measuring of entropy an information, In proceeding of the 4rth Barkley symposium on mathematics of statistics and probability, vol 1, pp. 547-561, 1961.
[18] C.E. Shannon, W. Weaver, The Mathematical theory of communication, The University of Illinois Press Urbana, 1964.
[19] L.S. Rompolis, Retrieving risk neutral densities from European option prices based on the principle of maximum entropy, Journal of Empirical Finance, vol 17, pp. 918-937, 2010.