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SUFFICIENT OPTIMALITY CONDITIONS FOR MULTI – OBJECTIVE PROGRAMMING PROBLEMS WITH BEN-TAL ALGEBRAIC OPERATIONS

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1. PRELIMINARIES AND RELATED RESULTS

Let $\mathbb{R}^n$ be the *n*-dimensional Euclidian space, and $\mathbb{R}, \mathbb{R}_+$ the sets of all real numbers and nonnegative numbers, respectively. Throughout this paper, the following convention for vectors in $\mathbb{R}^n$ will be followed:

$x < y$ if and only if $x_i < y_i, i = 1,2, \dots, n,$
$x \leqq y$if and only if $x_i < y_i, i = 1,2, \dots, n,$
$x \leq y$ if and only if $x_i \leqq y_i, i = 1,2, \dots, n,$ but $x \neq y$
$x > y$ is the negation of $x < y$.

Now, let us recall generalized operations of addition and multiplication introduced by Ben-Tal.

1) Let *h* be an *n* vector-valued continuous function, defined on $\mathbb{R}^n$ and possessing an inverse function h^{-1}. Define the *h*-vector addition of $x, y \in \mathbb{R}^n$ as

$x \oplus y = h^{-1}\big(h(x) + h(y)\big)$

and the *h* scalar multiplication of $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ as

$\alpha \otimes x = h^{-1}\big(\alpha h(x)\big).$

2) Let φ be a real-valued ontinuous function, defined on $\mathbb{R}$ and possessing an inverse function φ^{-1}. Then, the φ-addition of two numbers, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, is given by

$\alpha[+]\beta = \varphi^{-1}\big(\varphi(\alpha) + \varphi(\beta)\big),$

and the φ-scalar multiplication of $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ as

$\beta[\cdot]\alpha = \varphi^{-1}\big(\beta\varphi(\alpha)\big).$

Denote

$\oplus_{i=1}^{m} x^i = x^1 \oplus x^2 \oplus \dots \oplus x^m,\ x^i \in \mathbb{R}^n, i = 1,2, \dots, m,$
$[\sum_{i=1}^{m} \alpha_i] = \alpha_1[+]\alpha_2[+] \dots [+]\alpha_m, i = 1,2, \dots, m,$
$\alpha[-]\beta = \alpha[+]((-1)[\cdot]\beta, \alpha, \beta \in \mathbb{R}$

In the above Ben-Tal generalized algebraic operations, it is worth nothing that $\beta[\cdot]\alpha$ may not be equal to $\alpha[\cdot]\beta$ for $\alpha, \beta \in \mathbb{R}$. In addition it is clear that $1 \otimes x = x$ for any $x \in \mathbb{R}^n$ and $1[\cdot]\alpha = \alpha$ for any $\alpha \in \mathbb{R}$. For $\alpha, \beta \in \mathbb{R}$and $x \in \mathbb{R}^n$, the following conclusions can be obtained with easy

$$\varphi(\alpha[\cdot]\beta) = \alpha\varphi(\beta), h(\alpha \otimes x) = \alpha h(x),$$

$\alpha[-]\beta = \varphi^{-1}(\varphi(\alpha) - \varphi(\beta)).$

Avriel introduced the following concept, which plays an important role in our paper.

Definition 2.1 Let *f* be a real-valued function defined on $\mathbb{R}^n$, denote $\hat{f}(t) = \varphi\left(f\big(h^{-1}(t)\big)\right), t \in \mathbb{R}^n$.

For simplicity, write $\hat{f}(t) = \varphi f h^{-1}(t), t \in \mathbb{R}^n$. The function *f* is said to be (h, φ) –differentiable at $x \in \mathbb{R}^n$if $\hat{f}(t)$is differentiable at $t = h(x)$, and denoted by $\nabla^* f(x) = h^{-1}\left(\nabla\hat{f}(t)\big|_{t=h(x)}\right)$. In addition, it is said that *f* is (h, φ) – differentiable on $X \subset \mathbb{R}^n$ if it is (h, φ)– differentiable at each $x \in X$. A vector valued function is called (h, φ)– differentiable on $X \subset \mathbb{R}^n$if each of its components is (h, φ)– differentiableon *X*.

If *f* is differentiable at *x*, then *f* is (h, φ) – differentiable at *x*. We obtain this fact by setting *h* and φ are identity functions, respectively. However, the converse is not true.

Definition 2.2 Let *X* be a nonempty subset of$\mathbb{R}^n$ a functional $F: X \times X \times \mathbb{R}^n$is called (h, φ)– sublinear if for any *x*, $\bar{x} \in X$,

$F(x, \bar{x}; a_1 + a_2) \leqq F(x, \bar{x}; a_1)[+]F(x, \bar{x}; a_2), \forall\, a_1, a_2 \in \mathbb{R}^n,$
$F(x, \bar{x}; \alpha \otimes a) \leqq \alpha[\cdot]F(x, \bar{x}; a), \forall\, a \in \mathbb{R}^n, \alpha \geqq 0,$

From the above efinition, we can easy obtain that if F is a (h, φ) –sublinear functional then

$F(x, \bar{x}; \oplus_{i=1}^{m} a_i) \leqq [\sum_{i=1}^{m}] F\big(x, \bar{x}; a_{i,}\big), a_i \in \mathbb{R}^n, i = 1, \dots, m$

We collect the following properties of Ben-Tal generalized algebraic operations and (h, φ) – differentiable functions from literature, which will be used in the squeal.

Lemma 2.1 suppose that *f* is a real – valued function defined on $\mathbb{R}^n$, and (h, φ) – differentiable at$\bar{x} \in \mathbb{R}^n$. Then, the following statements hold:

(a) Let $x_{i,} \in \mathbb{R}^n, \lambda_i \in \mathbb{R}, i = 1,2, \dots, m$. Then
$\oplus_{i=1}^{m}(\lambda_i \otimes x^i) = h^{-1}(\sum_{i=1}^{m} \lambda_i h(x^i))$, $\oplus_{i=1}^{m} x^i = h^{-1}(\sum_{i=1}^{m} h(x^i))$.
(b) Let$\mu_i, \alpha_i \in \mathbb{R}$, i=1,2,...,m. Then
$\sum_{i=1}^{m}(\mu_i\varphi(\alpha_i)) = \varphi^{-1}(\sum_{i=1}^{m} \mu_i\varphi(\alpha_i))$, $[\sum_{i=1}^{m} \alpha_i] = \varphi^{-1}(\sum_{i=1}^{m} \varphi(\alpha_i))$.
(c) For$\alpha \in \mathbb{R}, \alpha\, [\cdot]f$ is (h, φ)– differentiable at $\bar{x}$ and $\nabla^*\big(\alpha[\cdot]f(\bar{x})\big) = \alpha \otimes \nabla^* f(\bar{x})$.

We need more properties of Ben-Tal generalized algebraic operations.

Lemma 2.2 Let i=1,2,...,m. The following statements hold:
(a) For $\alpha,\beta,\gamma \in \mathbb{R}$, then $\alpha[\cdot](\beta[\cdot]\gamma) = \beta[\cdot](\alpha[\cdot]\gamma) = (\alpha\beta)[\cdot]\gamma$
(b) For $\beta,\alpha_i \in \mathbb{R}$, then $\beta[\cdot][\sum_{i=1}^{m}\alpha_i] = \sum_{i=1}^{m}(\beta[\cdot]\alpha_i)$.
(c) For $\alpha,\beta,\gamma \in \mathbb{R}$, then $\gamma[\cdot](\alpha[-]\beta) = (\gamma[\cdot]\alpha)[-](\gamma[\cdot]\beta)$.
(d) For α_i, $\beta_i \in \mathbb{R}$ then $\sum_{i=1}^{m}(\alpha[-]\beta) = \sum_{i=1}^{m}\alpha_i\,[-]\sum_{i=1}^{m}\beta_i$,

$$\sum_{i=1}^{m}(\alpha[+]\beta) = \sum_{i=1}^{m}\alpha_i\,[+]\sum_{i=1}^{m}\beta_i$$

Lemma 2.3 Suppose that function φ, apears in Ben-Tal generalized algebraic operations, is strictly monotone with$\varphi(0) = 0$. Then, the following statements hold:
(a) Let $\gamma \geqq 0, \alpha,\beta,\gamma \in \mathbb{R}$, and $\alpha \leqq \beta$. Then $\gamma[\cdot]\alpha \leqq \gamma[\cdot]\beta$.
(b) Let $\gamma \geqq 0, \alpha,\beta,\gamma \in \mathbb{R}$, and $\alpha < \beta$. Then $\gamma[\cdot]\alpha \leqq \gamma[\cdot]\beta$.
(c) Let $\gamma > 0, \alpha,\beta,\gamma \in \mathbb{R}$, and $\alpha < \beta$. Then $\gamma[\cdot]\alpha < \gamma[\cdot]\beta$.
(d) Let $\gamma < 0, \alpha,\beta,\gamma \in \mathbb{R}$, and $\alpha \geqq \beta$. Then $\gamma[\cdot]\alpha \leqq \gamma[\cdot]\beta$.
(e) Let $\alpha_i,\beta_i \in \mathbb{R}$, i∈ M={1,2,...,m}. If $\alpha_i \leqq \beta_i$, for any $i \in M$, then
$\sum_{i=1}^{m}\alpha_i \leqq \sum_{i=1}^{m}\beta_i$.
If $\alpha_i \leqq \beta_i$for any $i \in M$and there exists at least an index $k \in M$such that
$\alpha_k < \beta_k$then
$\sum_{i=1}^{m}\alpha_i < \sum_{i=1}^{m}\beta_i$.

Lemma 2.4 Suppose that φ is a continuous one-to-one strictly monotone and onto function with $\varphi(0)$=0. Let$\alpha,\beta \in \mathbb{R}$. Then

$\alpha < \beta \Leftrightarrow \alpha[-]\beta < 0$

$\alpha \leqq \beta \Leftrightarrow \alpha[-]\beta \leqq 0$

$\alpha[+]\beta < 0 \Rightarrow \alpha < (-1)[\cdot]\beta$

$\alpha[+]\beta < 0 \Rightarrow \alpha < (-1)[\cdot]\beta$
$\alpha[+]\beta \leqq 0 \Rightarrow \alpha \leqq (-1)[\cdot]\beta$

Throughout this paper, we further assume that h is a continuous one-to-one and onto function with $h(0) = 0$. Similarly, suppose that φ is a continuous one-to-one strictly monotone and onto function with $\varphi(0) = 0$.
Under the above assumptions, it is clear that 0$[\cdot]\alpha = \alpha[\cdot]$0=0, for any $\alpha \in \mathbb{R}$.
Let X be a nonempty subset of $\mathbb{R}^n$, $C: X \times X \times \mathbb{R}^n \to \mathbb{R}$ is (h,φ)– convex and the functions:
$f = (f_1,\dots,f_k): X \to \mathbb{R}^k$ and $a = (a_1,\dots,a_m): X \to \mathbb{R}^m$ are (h,φ) – differentiable on the set X, with respect to the same (h,φ). Let $\rho = (\rho^1,\rho^2)$, where $\rho^1 = (\rho_1^1,\dots,\rho_k^1) \in \mathbb{R}^k$, $\rho^2 = (\rho_1^2,\dots,\rho_m^2) \in \mathbb{R}^m$.
Let $\theta(\cdot,\cdot): X \times X \to \mathbb{R}$.
Consider the following multi-objective programming problem:
$(P_1)_{h,\varphi}$: min $f(x) = (f_1(x),\dots,f_k(x)), x \in X \subset \mathbb{R}^n$, such that $a(x) \leqq 0$.
Let X_0 denote the feasible soltions for $(P_1)_{h,\varphi}$, assumed to be nonempty, that is:

$$X_0 = \{x \in X / a(x) \leqq 0\}$$

We denote $K = \{1,\dots,k\}$, $M = \{1,\dots,m\}$.
For $\bar{x} \in X_0$, we denote $M(\bar{x}) = \{j \in M, a_j(\bar{x}) = 0\}$.

Definition 2.3 For $i \in K, (f_i,a)$ is $(h,\varphi) - (C,\rho,\theta)$-type I at$\bar{x} \in X$ if for all $x \in X_0$, such that

$f_i(x)[-]f_i(\bar{x}) \geqq C_{x,\bar{x}}(\nabla^* f_i(\bar{x}))[+]\left(\rho_i^1[\cdot]\theta(x,\bar{x})\right), i \in M$

and

$(-1)[\cdot]a_j(\bar{x}) \geqq C_{x,\bar{x}}\left(\nabla^* a_j(\bar{x})\right)[+]\left(\rho_j^2[\cdot]\theta(x,\bar{x})\right), j \in M$

In the above definition $x \neq \bar{x}$ and (16) is a strict inequality, then we say that (f_i,a) is semi-strictly $(h,\varphi) - (C,\rho,\theta)$-type I at $\bar{x}$.

Definition 2.4 For $i \in K, (f_i,a)$ is said to be quasi $(h,\varphi) - (C,\rho,\theta)$-type I at $\bar{x} \in X$ if for all $x \in C$ such that

$$\sum_{i=1}^{k} f_i(x) \leqq \sum_{i=1}^{k} f_i(\bar{x}) \Rightarrow$$

$$\sum_{i=1}^{k} C_{x,\bar{x}}(\nabla^* f_i(\bar{x}))\,[+]\sum_{i=1}^{k}(\rho_i^1)[\cdot]\ \theta(x,\bar{x}) \leqq 0,$$

and

$$(-1)[\cdot]\sum_{j=1}^{p} a_j\,(\bar{x}) \leqq 0 \Rightarrow$$
$$\sum_{j=1}^{p} C_{x,\bar{x}}\big(\nabla^* a_i(\bar{x})\big)\,[+]\sum_{j=1}^{p}\big(\rho_j^2\big)[\cdot]\ \theta(x,\bar{x}) \leqq 0.$$

Definition 2.5 For $i \in K, (f_i, a)$ is said to be pseudo $(h,\varphi)-(C,\rho,\theta)$-type I at $\bar{x} \in X$ if for all $x \in C$ such that

$$\sum_{i=1}^{k} C_{x,\bar{x}}\big(\nabla^* f_i(\bar{x})\big)\,[+]\sum_{i=1}^{k}(\rho_i^1)[\cdot]\ \theta(x,\bar{x}) \geqq 0 \Rightarrow$$
$$\sum_{i=1}^{k} f_i(x) \leqq \sum_{i=1}^{k} f_i(\bar{x})$$

and

$$\sum_{j=1}^{p} C_{x,\bar{x}}\big(\nabla^* a_i(\bar{x})\big)\,[+]\sum_{j=1}^{p}\big(\rho_j^2\big)[\cdot]\ \theta(x,\bar{x}) \geqq 0 \Rightarrow$$
$$(-1)[\cdot]\textstyle\sum_{j=1}^{p} a_j\,(\bar{x}) \geqq 0.$$

Definition 2.6 For $i \in K, (f_i, a)$ is said to be pseudo $(h,\varphi)-(C,\rho,\theta)$-type I at $\bar{x} \in X$ if for all $x \in C$ such that

$$\sum_{i=1}^{k} f_i(x) \leqq \sum_{i=1}^{k} f_i(\bar{x}) \Rightarrow$$
$$\sum_{i=1}^{k} C_{x,\bar{x}}\big(\nabla^* f_i(\bar{x})\big)\,[+]\sum_{i=1}^{k}(\rho_i^1)[\cdot]\ \theta(x,\bar{x}) \leqq 0,$$

and

$$\sum_{j=1}^{p} C_{x,\bar{x}}\big(\nabla^* a_i(\bar{x})\big)\,[+]\sum_{j=1}^{p}\big(\rho_j^2\big)[\cdot]\ \theta(x,\bar{x}) \geqq 0 \Rightarrow$$
$$(-1)[\cdot]\textstyle\sum_{j=1}^{p} a_j\,(\bar{x}) \geqq 0.$$

Definition 2.7 For $i \in K, (f_i, a)$ is said to be pseudo quasi $(h,\varphi)-(C,\rho,\theta)$-type I at $\bar{x} \in X$ if for all $x \in C$ such that

$$\sum_{i=1}^{k} C_{x,\bar{x}}\big(\nabla^* f_i(\bar{x})\big)\,[+]\sum_{i=1}^{k}(\rho_i^1)[\cdot]\ \theta(x,\bar{x}) \geqq 0 \Rightarrow$$
$$\sum_{i=1}^{k} f_i(x) \leqq \sum_{i=1}^{k} f_i(\bar{x})$$

and

$$(-1)[\cdot]\sum_{j=1}^{p} a_j\,(\bar{x}) \leqq 0 \Rightarrow$$
$$\sum_{j=1}^{p} C_{x,\bar{x}}\big(\nabla^* a_i(\bar{x})\big)\,[+]\sum_{j=1}^{p}\big(\rho_j^2\big)[\cdot]\ \theta(x,\bar{x}) \leqq 0.$$

3. SUFFICIENT OPTIMALITY CONDITIONS

In this section, we establish sufficient optimality conditions for a feasible solution $\bar{x}$ to be a weak minimum for $(P_1)_{h,\varphi}$, under the $(h,\varphi)-(C,\rho,\theta)$-type I and pseudo quasi $(h,\varphi)-(C,\rho,\theta)$-type I assumptions.

Theorem 3.1 Suppose that there exist a feasible solution $\bar{x} \in C$ and $\bar{\lambda} = (\overline{\lambda_1}, \ldots, \overline{\lambda_k}) \in \mathbb{R}^k$, $\bar{\lambda} \geq 0, \overline{\mu_j} \geqq 0, j \in M(\bar{x})$, $\sum_{i=1}^{k}\overline{\lambda_i}+\sum_{j=1}^{m}\overline{\mu_j} = 1$ such that

$\left(\oplus_{i=1}^{k}\,(\overline{\lambda_i} \otimes \nabla^* f_i(\bar{x}))\right) \oplus \left(\oplus_{j\in M(x)}\left(\overline{\mu_j} \otimes \nabla^* a_j(\bar{x})\right)\right)$=0.

If for $i \in K, \left(f_i, g_{J(x)}\right)$ is $(h,\varphi)-(C,\rho,\theta)$-type I at $\bar{x}$ with

$\left(\sum_{i=1}^{k}\left(\overline{\lambda_i}\rho_i^1\right)[\cdot]\,\theta(\cdot,\bar{x})\right)[+]\left(\sum_{j\in J(\bar{x})}(\overline{\mu_j}\rho_j^2)\,[\cdot](\theta(\cdot,\bar{x}))\right) \geqq 0,$
where $a_{J(\bar{x})} = (a_j)_{j\in J(\bar{x})}$. Then $\bar{x}$ is a weak minimum for $(P_1)_{h,\varphi}$.

Theorem 3.2 Suppose that there exist $\bar{x} \in C$ and $\bar{\lambda} = \left(\bar{\lambda}_1, \dots, \bar{\lambda}_k\right) \in \mathbb{R}^k$,
$\bar{\lambda} \geq 0, \overline{\mu_j} \geqq 0, j \in M(\bar{x})$.
If for $i \in K, \left(\overline{\lambda_i}[\cdot]f_i, \bar{G}_{J(x)}\right)$ is pseudo quasi $(h,\varphi) - (C,\rho,\theta)$-type I at $\bar{x}$ with
$\left(\sum_{i=1}^{k}(\rho_i^1)[\cdot]\,\theta(\cdot,\bar{x})\right)[+]\left(\sum_{j\in J(\bar{x})}(\rho_j^2)\,[\cdot](\theta(\cdot,\bar{x}))\right) \geqq 0,$
Where $\bar{G}_{J(x)} = (\bar{G}_j)_{j\in J(x)}, \bar{G}_j = \overline{\mu_j}[\cdot]g_j$, then $\bar{x}$ is a weak minimum for $(P_1)_{h,\varphi}$.

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