"Mircea cel Batran" Naval Academy Scientific Bulletin, Volume XVIII – 2015 – Issue 2
Published by "Mircea cel Batran" Naval Academy Press, Constanta, Romania // The journal is indexed in:
PROQUEST SciTech Journals, PROQUEST Engineering Journals, PROQUEST Illustrata: Technology, PROQUEST Technology Journals, PROQUEST Military Collection PROQUEST Advanced Technologies & Aerospace

TIME, LINEAR AND LEXICOGRAPHIC TRANSPORTATION PROBLEM WITH IMPURITIES

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Abstract: *An extension of the time transportation problem is considered when the goods may have some impurities and the final quantity of arrived goods reaching to the destination must have certain specifications. This time transportation problem is in relation with the lexicographical linear transportation problem with impurities. An algorithm is presented to solve this problem, taking into account the directions given by Isermann.*
Keywords*: transportation problem, impurities, bottleneck*
Mathematics Subject Classification 2000: 90B06, 90C08

INTRODUCTION

The problems of transport with impurities in goods are in fact very important, due to the high frequency of such problem. In many cases in real life, the impurities have special characteristics according to the source and request of the receivers. Such transportation problems of minimizing the cost with objective linear functions can be efficiently solved with the algorithm of Haley and Smith, 1966.
Chandra and Saxena in 1983, using the results of duality for fractional programming, they modified the algorithm of Haley and Smith (1966) to solve the transportation fractional problem after minimizing the cost in the impurities problem.
Next, we study the bottleneck transportation problem with impurities. This problem is encountered in connection with the transport of the perishables goods, maximum emergency situations or when the military equipment or the units' battles are sent from their base on the battle field.
The bottleneck transportation problems were already studied by Hammer in 1969, 1971, Szwarc in 1966, 1971, Garfinkel and Rao in 1971 and Isermann in 1984, but they didn't take into consideration the impurities in goods.
Next, is presented an algorithm for solving the bottleneck transportation problem, making a connection of this with the lexicographical linear transportation problem with impurities (L.T.P.) and integrating the results of Haley and Smith (1966), Chandra and Saxena (1983) and also of Isermann (1984).
This algorithm takes into consideration the special structure of the transportation problem and strongly depends on the optimality conditions which are similar with the optimality conditions of those given by Isermann in 1984, but will appear some modifications because of the presence of the impurities.

1. THE PROBLEM FORMULATION

The mathematical formulation of the problem is the following:

$$\text{(T.P.)} \qquad Min\ \mathrm{t} = \max_{i,j}\left\{t_{ij} \,\middle|\, x_{ij} > 0\right\} \tag{1}$$

with the restrictions:

$$\sum_j x_{ij} = a_i\ , \tag{2}$$

$$\sum_i x_{ij} = b_j\ , \tag{3}$$

$$\sum_i f_{ijk} \cdot x_{ij} \le q_{jk}\ , \tag{4}$$

$$x_{ij} \ge 0\ , \tag{5}$$

$(i = 1,2,...,M)$, $j = 1,2,...,N$, $k = 1,2,...,P$

where a_i , b_j , x_{ij} , t_{ij} are the classical known notations,

f_{ijk} = the unit from the k impurity $(k = 1,2,...,P)$ belonging to transported goods from source i , to destination j .

q_{jk} = the impurities quantity of type k , accepted on destination j .

In this problem a_i and b_j are considered nonnegative numbers and $\sum_{i=1}^{M} a_i = \sum_{j=1}^{N} b_i$.

Without restrictions (4) considering the impurities, the problem (T.P) becomes a usual bottleneck transportation problem studied by Hammer in 1969 and Isermann in 1984.

In this context, the restrictions (4) are written as follows:

$$\sum_i f_{ijk} \cdot x_{ij} + x_{N+k,j} = q_{jk} \qquad (6)$$

$$x_{N+k,j} \geq 0 \qquad (7)$$

where $x_{N+k,j}$ are variables of compensations to impurities restrictions and an admissible base solution will consist in $NP + M + N - 1$ base variables.

We associate to the (T.P.) problem the next lexicographical transportation problem with impurities:

$$Lex \min Z = \sum_i \sum_j d_{ij} x_{ij} \qquad (8)$$

with restrictions (2) $\div$ (5)

The above formulation is made partitioning the set $\gamma = M \times N$ into the subsets $\gamma_c (c = 1,2,...,e)$, like (Isermann, 1984).

Any of the subsets γ_c will consist in the pairs $(i, j) \in \gamma$, for which the transportation times t_{ij} have the same numerical value.

The subset γ_1 contains all the pairs $(i, j) \in \gamma$, with t_{ij} the greatest, γ_2 contains all the pairs $(i, j) \in \gamma$, with t_{ij} taking the next bigger value, etc. Therefore, the subset γ_e contains all the pairs $(i, j) \in \gamma$, with t_{ij} taking the smallest value. After that, to each value x_{ij} with $(i, j) \in \gamma$, $(c = 1,2,...,e)$, is associated a unitary vector of dimension $e \times 1$ and we consider the vectors $d_{ij} := e_c$ if $(i, j) \in \gamma_c$.

2. DUALITY AND OPTIMALITY CONDITIONS

We consider the vector variables $u_i \left(i = \overline{1, M}\right)$, $v_j \left(j = \overline{1, N}\right)$ and $t_{jk} \left(j = \overline{1, N};\ k = \overline{1, P}\right)$ defined as follows:

$$d_{ij} - \left(u_i + \mathrm{v}_j - \sum_k t_{jk} f_{ijk} \right) = 0 \qquad (9)$$

for those i, j for which x_{ij} is in base and:

$$t_{jk} = 0 \qquad (10)$$

for those j and k for which $x_{N+k,j}$ is in base.

Let $\hat{U} = \left(\hat{u}_1, \hat{u}_2, ..., \hat{u}_M ;\ \hat{\mathrm{v}}_1, \hat{\mathrm{v}}_2, ..., \hat{\mathrm{v}}_N ;\ \hat{t}_{11}, \hat{t}_{12}, ..., \hat{t}_{NP}\right)$ be a solution of (9) and (10). The vectors u_i, v_j and t_{jk} introduced above are the variables from the dual problem of the primal problem (L.T.P).

The dual of the problem (L.T.P) defined by (8), (2), (3), (5), (6), (7) can be written as follows:

$$\text{(D.L.T.P)} \quad Lex \max G = \sum_i u_i a_i + \sum_j \mathrm{v}_j b_j + \sum_j \sum_k t_{jk} q_{jk} \qquad (11)$$

with restrictions:

$$u_i + \mathrm{v}_j + \sum_k f_{ijk} t_{jk} \leq d_{ij} \qquad (12)$$

$$t_{jk} \leq 0 \qquad (13)$$

and:

u_i, v_j and t_{jk} without sign restrictions. (14)

$\left(i = 1, 2, ..., M;\ j = 1, ..., N;\ k = 1, ..., P \right)$.

Now, like Haley and Smith in 1966 as well as Chandra and Saxena in 1983, we consider the comparative differences.

$$\Delta_{ij} = \left(d_{ij} - u_i - \mathrm{v}_j - \sum_k f_{ijk} t_{jk} \right) > 0 \tag{15}$$

and:

$$\Delta_{N+k,j} = t_{jk} = 0 \text{, for } x_{N+k,j} > 0 \tag{16}$$

Then, using the duality theory between (L.T.P) and (D.L.T.P) in the absence of degeneration, the optimality criterions are:

$$\Delta_{ij} = \left[d_{ij} - \hat{u}_i - \hat{\mathrm{v}}_j - \sum_k \hat{t}_{jk} f_{ijk} \right] \geq 0 \tag{17}$$

and:

$$\Delta_{N+k,j} = \hat{t}_{jk} \geq 0 \text{, for those } (i,j) \in \gamma . \tag{18}$$

3. THE ALGORITHM

In this section, an algorithm in presented to determine an admissible optimal base solution of the problem (L.T.P) into a finite number of iterations.
The steps of the algorithm are:

Step 1

The lower bound t_e of t has to be determined, (according to Garfinkel and Rao, 1971) to reduce the dimension of the vectors d_{ij} in the problem (8). Now, we have $t_e > t_{ij}$ for at least a pair $(i,j) \in \gamma$. Here, γ_c contain all the pairs $(i,j) \in \gamma$ with $t_e > t_{ij}$.

Step 2

X^1 has to be determined as an admissible initial base solution of the matrix $T = \lfloor t_{ij} \rfloor$ applying the method of Haley and Smith (1966).

Step 3

From the bottleneck time t of the solution X^1, an upper bound t_u has to be determined . Let $t_u < t_{ij}$ for at least a pair $(i,j) \in \gamma$.

Step 4

The set is partitioning $\gamma = M \times N$, into the subsets γ_c and the vectors are determined $d_{ij} := e_c$ for all $(i,j) \in \gamma$, to obtain the matrix of cost D.

Step 5

Using the solution X^1 the associated multipliers are recursively calculated (the dual variables) u_i, v_j and t_{jk} defined as follows:

$$d_{ij} - \left(u_i + \mathrm{v}_j + \sum_k t_{jk} f_{ijk} \right) = 0 \tag{19}$$

for those i, j for which x_{ij} is in the base:

$$t_{jk} = 0 \tag{20}$$

for those j, k for which $x_{N+k,j}$ is not in the base.

Let $\hat{U} = \left(\hat{u}_1, \hat{u}_2, ..., \hat{u}_M ;\ \hat{\mathrm{v}}_1, \hat{\mathrm{v}}_2, ..., \hat{\mathrm{v}}_{\mathrm{P}} ;\ \hat{t}_{jk}, ..., \hat{t}_{NP} \right)$ be a solution of (19) and (20).

Step 6

The criterion vectors are evaluated:

$$\Delta_{ij} = \left[d_{ij} - \left(\hat{u}_i + \hat{\mathrm{v}}_j \right) + \sum_k \hat{t}_{jk} f_{ijk} \right] \tag{21}$$

$$\Delta_{N+k,j} = \hat{t}_{jk} \text{ for all the others from the outside of base.} \tag{22}$$

Step 7

If all Δ_{ij} and $\Delta_{N+k,j}$ are lexicographically greater than or equal to zero, the current base solution is the optimal solution of the problem (L.T.P). Go to step 10. Contrary go to step 8.

Step 8

Choose:

$$\Delta_{gh} = Lex\min\left\{\Delta_{ij}\middle|\Delta_{ij} \le 0\right\} \tag{23}$$

or

$$\Delta_{N+k,j} = Lex\min\left\{\Delta_{N+k,j}\middle|\Delta_{N+k,j} \le 0\right\}.$$

Applying the selection rule (23), the variable x_{gh} or $x_{N+k,j}$ turns into a variable base of the new admissible base solution.

Step 9

To change the current solution into a new admissible base solution, add n_{gh} or $n_{N+k,j}$ to the variable x_{gh} or $x_{N+k,j}$ and n_{rs} or $n_{N+w,s}$. The quantities $"n"$ satisfy the equations:

$$\sum_{r=1}^{N} n_{rs} = 0,\ \left(s = 1,2,...,M\right) \tag{24}$$

$$\sum_{s=1}^{M} n_{rs} = 0,\ \left(r = 1,2,...,N\right) \tag{25}$$

$$\sum_{w=1}^{P} f_{rsw} \cdot n_{rs} + n_{N+w,s} = 0,\ \left(s = 1,2,...,M;\ \ w = 1,...,P\ \right) \tag{26}$$

with $n_{11} = 1$. Here $n_{rs} = 0$, if x_{rs} is not in the base and $n_{N+w,s} = 0$, if $x_{N+w,s}$ is not in the base. There is $N \cdot P + M + N - 1$ independent equations of (24), (25), (26) and there is $N \cdot P + M + N$ unknowns. Also, the variables values in the new admissible base solution are given by $x_{rs} + n_{rs} \cdot \theta$, $x_{N+w,s} + n_{N+w,s} \cdot \theta$.

Choosing a convenient value for θ, one of the variables can be brought down to zero, while the others remain positive and a new admissible base solution is obtained. The chosen value is:

$$\theta = \min_{\substack{n_{rs}<0 \\ n_{N+w,s}<0}} \left[-\frac{x_{rs}}{n_{rs}};-\frac{x_{N+w,s}}{n_{N+w,s}}\right] \tag{27}$$

Go to Step 5.

Step 10

If $\hat{X} = \left(\hat{x}_{ij}\right)$ is an optimal solution of the problem (L.T.P), then: $\hat{Z} = \sum_i \sum_j \left(d_{ij}\hat{x}_{ij}\right)$ and $\hat{c}$ are the clews of the first positive component of the optimal flux vector $\hat{Z}$. Also, $\hat{t} = t_{ij}$, with $\left(i,j\right) \in \gamma_c$ being the optimal time of the bottleneck transportation problem.

The optimal solution of transport $\hat{X} = \left(\hat{x}_{ij}\right)$ also minimize the linear function $z_{\hat{c}} = \sum_{i,j}\left(x_{ij}\right)$, $\left(i,j\right) \in \gamma_{\hat{c}}$ which express the total repartition which the time $\hat{t}$ requests.

4. NUMERICAL EXAMPLE

The algorithm is illustrated by the next numerical example which is similar with that given by Haley and Smith in 1966.

The data of the problem is given in the next table.

		The receivers centre				
		1	2	3	a_i	P_i
The source centre	1	4	2	5	7	0,4
	2	5	1	5	12	0,8
	3	6	8	3	6	0,7

	b_j	5	10	10		
	L_j	0,7	0,7	0,7		

P_i = bring in quantities $(i=1,2,3)$ from the send center;

L_j =the maximum quantity of admissible impurities to receivers center $(j=1,2,3)$.

Let x_{ij} be the transported quantities from i to j. Is requested to minimize $t=\max\limits_{i,j}\{t_{ij}|x_{ij}>0\}$ with restrictions:

$$\sum_{j=1}^{3} x_{ij}=a_i \ , \ (i=1,2,3),$$

$$\sum_{i=1}^{3} x_{ij}=b_j \ , \ (j=1,2,3),$$

$$\sum_{i=1}^{3} p_i x_{ij} \le L_j b_j \ ,$$

$$x_{ij} \ge 0 \ .$$

The lower bound t_e of the above problem can be obtained by calculating the threshold on line and column and we obtain: 2, 5, 3, respectively 4, 1, 5. Therefore, the minimal threshold $t_e=\max\{2,5,3,4,1,5\}=5$. Applying the method of Haley and Smith from 1966 to the matrix $T=\lfloor t_{ij}\rfloor$, the admissible base solution X^1 is:

$$x_{11}=3,\ x_{12}=5/2,\ x_{13}=3/2,\ x_{22}=15/2, x_{23}=9/2,\ x_{31}=2,\ x_{33}=4,\ x_{41}=9 .$$

To this solution is corresponding the transportation time (bottleneck time) $t_u=6$. Therefore, $e=4$ and so γ, has four subsets :

$$\gamma_1:=\{(i,j)\in\gamma|t_{ij}>6\},\ \gamma_2:=\{(i,j)\in\gamma|t_{ij}=6\},$$
$$\gamma_3:=\{(i,j)\in\gamma|t_{ij}=5\},\ \gamma_4:=\{(i,j)\in\gamma|t_{ij}<5\}.$$

Now, the matrix of cost D of the lexicographical linear transportation problem with impurities is:

$$D=\begin{bmatrix} e_4 & e_4 & e_3 \\ e_3 & e_4 & e_3 \\ e_2 & e_1 & e_4 \end{bmatrix}.$$

Using X^1, the multipliers u_i, v_j and t_{jk} can be determined and then they can calculate Δ_{ij} and $\Delta_{N+k,j}$. Because x_{41} is in the base, $t_{11}=0$. Now, an arbitrary value is given to v_1 (for example, the value 0). Because x_{11} is in the base, we have $d_{11}=\mathrm{v}_1+u_1+4\cdot t_{11}$ which gives $u_1=e_4$.

Because, x_{13} and x_{33} are both in the base, we have:

$$d_{13}=\mathrm{v}_3+u_1+4\cdot t_{31}, \text{ therefore } 4\cdot t_{31}+\mathrm{v}_3=e_3-e_4,$$
$$d_{33}=\mathrm{v}_3+u_3+7\cdot t_{31}, \text{ therefore } 7\cdot t_{31}+\mathrm{v}_3=e_4-e_2,$$

which leads to:

$$\begin{cases} t_{31}=-1/3\cdot e_2-1/3\cdot e_3+2/3\cdot e_4 \\ v_3=4/3\cdot e_2+7/3\cdot e_3-11/3\cdot e_4 \ . \end{cases}$$

Using a similar way for the base variables x_{31}, x_{23} and x_{22} we obtain:

$$u_3=e_2,\ u_2=4/3\cdot e_2+4/3\cdot e_3-5/3\cdot e_4,$$
$$t_{21}=-1/3\cdot e_2-1/3\cdot e_3+2/3\cdot e_4, \mathrm{v}_2=-4/3\cdot e_2+4/3\cdot e_3-8/3\cdot e_4.$$

Having determined the dual variables, the values of Δ_{ij} and $\Delta_{N+k,j}$ are obtained like this:

$$\Delta_{21}=-4/3\cdot e_2-1/3\cdot e_3+5/3\cdot e_4,\ \Delta_{32}=e_1+e_3-2e_4,$$

$$\Delta_{42} = -1/3 \cdot e_2 - 1/3 \cdot e_3 + 2/3 \cdot e_4 \,,\, \Delta_{43} = -1/3 \cdot e_2 - 1/3 \cdot e_3 + 2/3 \cdot e_4 .$$

From the flux vector $Z(X^1) = (0,2,6,17)^T$ is obtained for the current solution X^1, the transportation time (bottleneck time) is 6 and the flux of bottleneck is 2.

Because X^1 is not optimal, we apply the selection rule of the Step 8, the variable x_{21} will come into the base and so n_{21} is added to this variable and n_{rs} is added to all base variables x_{rs}.

The quantities "n" satisfies the equations (24), (25) and (26) and can be determined in the following manner: here $n_{32} = 0$, $n_{42} = 0$, $n_{43} = 0$, because x_{32}, x_{42} and x_{13} are not in the base. Now, for $j = 2$, $n_{12} + n_{22} + n_{32} = 0$ and $4n_{12} + 8n_{22} + 7n_{32} + n_{42} = 0$, we obtain $n_{12} = 0$, $n_{22} = 0$. For $i = 2$, $n_{12} + n_{22} + n_{23} = 0$, we obtain $n_{23} = -n_{21}$. For $j = 3$, $n_{13} + n_{23} + n_{33} = 0$ and $4n_{13} + 8n_{23} + 7n_{33} + n_{43} = 0$, we obtain $n_{33} = 4/3 \cdot n_{21}$. For $i = 3$, $n_{11} + n_{21} + n_{31} = 0$, we obtain $n_{31} = -4/3 \cdot n_{21}$. For $j = 1$, $4n_{11} + 8n_{21} + 7n_{31} = 0$, we obtain $n_{11} = 1/3 \cdot n_{21}$, $n_{41} = 0$. For $i = 1$, $n_{11} + n_{12} + n_{13} = 0$, we obtain $n_{13} = -1/3 \cdot n_{21}$.

Therefore, $n_{21} = Min\left[3/2 \cdot 1/3,\ 9/2 \cdot 1,\ 2 \cdot 3/4\right] = 3/2$.

Using this value of n_{21}, we obtain the new admissible base solution:

$$x_{11} = 3 + 1/3 \cdot 3/2 = 7/2 \,,\ x_{12} = 5/2 \,,\ x_{13} = 3/2 - 1/3 \cdot 3/2 = 1 \,,\ x_{21} = 3/2 .$$

Similarly, $x_{22} = 15/2$, $x_{23} = 3$, $x_{31} = 0$, $x_{33} = 6$, $x_{41} = 9$.

Proceeding like above, the values for some iteration are the following: the admissible initial base solution is:

$$X^1 = \begin{bmatrix} 3 & 5/2 & 3/2 \\ 0 & 15/2 & 9/2 \\ 2 & 0 & 4 \\ \cdots & \cdots & \cdots \\ 9 & 0 & 0 \end{bmatrix};$$

the transportation time (bottleneck time) is $t = 6$ and the transportation flux (bottleneck flux) is 2.

After the first iteration, the admissible base solution is:

$$X^2 = \begin{bmatrix} 7/2 & 5/2 & 1 \\ 3/2 & 15/2 & 3 \\ 0 & 0 & 6 \\ \cdots & \cdots & \cdots \\ 9 & 0 & 0 \end{bmatrix};$$

the transportation time (bottleneck time) is $t = 6$ and the transportation flux (bottleneck flux) is $11/2$.

After the second iteration the admissible base solution is:

$$X^3 = \begin{bmatrix} 5/4 & 5/2 & 13/4 \\ 15/4 & 15/2 & 3/4 \\ 0 & 0 & 6 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 9 \end{bmatrix};$$

the transportation time (bottleneck time) is $\hat{t} = 5$ and the transportation flux (bottleneck flux) is $31/4$, because all Δ_{ij} are lexicographically greater than 0. Therefore, X^3 is the optimal solution of the problem (L.T.P). The optimal value of the flux vector Z is $Z(\hat{X}^3) = (0,0,31/4,69/4)^{\mathrm{T}}$. Thus, the optimal transport time (bottleneck time) is $\hat{t} = 5$ and the transportation flux (bottleneck flux) is $31/4$.

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