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ABOUT SOME KKT-TYPE RESULTS IN LOCALLY CONVEX CONES

Ligia-Adriana SPORIS[1]
Traian FLOREA[2]
Corneliu MOROIANU[3]
[1] Lecturer PhD, Dep. IMAN, "Mircea cel Batran" Naval Academy, Constanta, 1 Fulgerului Street
[2] Prof. PhD, Dep. IMAN, "Mircea cel Batran" Naval Academy, Constanta, 1 Fulgerului Street
[3] Associate Prof. PhD, Dep. IMAN, "Mircea cel Batran" Naval Academy, Constanta, 1 Fulgerului Street

Abstract: *The aim of this note is to present some Korovkin type-approximation results in locally convex cones.*
Key words: *locally convex cones, Korovkin - closure*

Introduction

• Positive approximation processes play a fundamental role in Approximation Theory and they appear in a natural manner in qualitative aspects problems of continuous functions approximation.

• Starting with the study of key role of Bernstein polynomials in the proof of Weierstrass Theorem, P.P.Korovkin published in 1953 his celebrated theorem on the convergence of sequences of positive linear operators (i.e. $(T_n)_n$ is a positive approximation process on $C([0,1])$ if $(T_n)_n$ uniformly on $[0,1]$ converges towards identity for the test functions of $S_0=\{1,x,x^2\}$.

He also proved that:

a) every test set (or Korovkin set) (i.e. a set S which replaces S_0 in the previous text) has to contain at least three elements:

b) in addition, a triple set $S=\{f_0,f_1,f_2\}$ forms a test set exactly when it is a three-degree Cebîşev system (i.e. every nonzero linear combination vanishes in at last two points).

• The simplicity and the power of Korovkin classical theorem raised the interest of a large number of mathematicians, since '50, to study it and to obtain numerous extensions and generalizations.

• So, all these results appeared determined to get rise a new direction in mathematical research: „Korovkin type Approximations Theory" (shortly, KAT). Also, many mathematicians were interested by the links between KAT and others fields of mathematical research: classical Approximations Theory, Functional Analysis (convexity theory, convergence of sequences of positive linear operators on: Banach lattices, Banach algebras, C*-algebras, approximation in functions algebras), Measure Theory, Probabilistic Theory (approximation processes generated by probabilistic methods, weak-convergence of positive Radon measures nets), approximation of Dirichlet type problems solutions.

• The most important moments in KAT-development are:

1. The '50 years are marqued by the results of Korovkin and his students, E.N.Morozov and V.J.Volkov.
2. In '60, we can remark the work of J.Šaškin, who gave a characterization of finite Korovkin sets in $C(X)$ (where X is a compact metric space).
3. Then, in '70 years the KAT-development continued with the research in $C_0(X)$, X local compact space, due to Wulbert, Berens, Lorentz, Donner and Bauer.
4. In '80, we assist to the KAT extension to abstract spaces. These studies were realized by the representants of Russian school: Krasnoselski, Lifschitz, Rubinov, Kutadeladze, Vasiliev and respectively of German school: Wolff, Scheffold, Flösser, Donner, Irmisch.
5. From '90 till now, an essential contribution is due to: Pannenberg, Beckoff, Labsker, Limaye and Italian school: Michelli, Altomare, Campiti, Romanelli and not at last, the Romanian mathematician from Technical University of Cluj-Napoca, Ioan Raşa.

• In KAT, the principal research directions are concerned with the characterization of Korovkin closures and the obtaining sufficiently and necessary conditions as a Korovkin closure to coincide with a certain subspace.

Preliminaries and notations

• We shall present an outline of the principal concepts and notation for this -note.

• Let be$(\boldsymbol{C},\boldsymbol{V})$, a locally convex-cone and$\boldsymbol{G}\subset\boldsymbol{C}$, a subcone. We introduce a linear pre-order relation on $\boldsymbol{C}^*$, of Choquet-type $\mu,\nu\in C^*,\ \mu\prec_G\nu\Leftrightarrow b\in G,\ \mu(b)\le\nu(b)$ (2.1).

• Def.2.1.: Let be $(\boldsymbol{C},\boldsymbol{V})$, a locally convex-cone and $\boldsymbol{G}\subset\boldsymbol{C}$, subcone. Let be $\boldsymbol{a}\in\boldsymbol{C}$ and $\boldsymbol{\mu}\in\boldsymbol{C}^*$.

a) $\boldsymbol{a}$ is named **a** $\boldsymbol{G}$-superharmonic in

$$\mu\Leftrightarrow\begin{cases}(i)\ \ \mu(\mathrm{a})<+\infty\\(ii)\ \ (\forall)\nu\in\mathrm{C}^*,\ \nu\prec_{\mathrm{G}}\mu\Rightarrow\nu(a)\le\mu(a).\end{cases}\quad(2.2)$$

$Sup_G(\mu)=\{\mathrm{a}\in\mathrm{C}\mid \mathrm{a},G$ - superharmonic in $\mu\}$.

b) $\boldsymbol{a}$ is named **a** $\boldsymbol{G}$-subharmonic in

$$\mu\Leftrightarrow\begin{cases}(i)\ \ \mu(\mathrm{a})<+\infty;\\(ii)\ \ (\forall)\nu\in\mathrm{C}^*,\ \mu\prec_{\mathrm{G}}\nu\Rightarrow\mu(a)\le\nu(a).\end{cases}\quad(2.3)$$

$Sub_G(\mu)=\{\mathrm{a}\in\mathrm{C}\mid \mathrm{a},G$ - subharmonic in $\mu\}$

c) $\boldsymbol{a}$ is called $\boldsymbol{G}$-harmonic in $\boldsymbol{\mu}\Leftrightarrow\boldsymbol{a}$ is $\boldsymbol{G}$-superharmonic and $\boldsymbol{G}$-subharmonic in $\boldsymbol{\mu}$.

$$A_G(\mu)\overset{not}{=}Sup_G(\mu)\cap Sub_G(\mu)$$

It easy to see that $\boldsymbol{Sup_G(\mu)}$ (resp. $\boldsymbol{Sub_G(\mu)}$) is a subcone, which contains $\boldsymbol{G}$ and it's closed in the symmetric topology. If $\boldsymbol{C}$ is $\boldsymbol{a}\wedge(\boldsymbol{v})$- semi-lattice and $\boldsymbol{\mu}$ is $\boldsymbol{bs}$ – directed there $\boldsymbol{Sup_G(\mu)}$ (resp. $\boldsymbol{Sub_G(\mu)}$) is $\boldsymbol{a}\wedge(\boldsymbol{v})$ semilattice.

• Def.2.2.: Let (C,V) be a locally convex cone and $G\subset C$, a subcone. Let $a\in C$.

a) $\hat{a}_v:C^*\to\overline{R}$,

$$\hat{a}_v(\mu)=\begin{cases}\inf\{\mu(b)\mid b\in G,\ \mathrm{a}\prec\mathrm{b}+\mathrm{v}\},\ \text{dacã }(\exists)\mathrm{b}\in\mathrm{G}\ \text{ a.i. }\ \mathrm{a}\prec\mathrm{b}+\mathrm{v},\mathrm{v}\in\mathrm{V}\\+\infty,\ \text{în rest.}\end{cases}$$

(2.4) is called the $\vee$ – superior envelope of the element $a\in C$.

b) $\breve{a}_v:C^*\to R\cup\{-\infty\}$,

$$\breve{a}_v(\mu)\begin{cases}\sup\{\mu(b)\mid b\in G,\ \mathrm{b}\prec\mathrm{a}+\mathrm{v}\},\ \text{dacã }(\exists)\mathrm{b}\in\mathrm{G}\ \text{ a.i. }\ \mathrm{b}\prec\mathrm{a}+\mathrm{v},\mathrm{v}\in\mathrm{V},\\-\infty,\ \text{în rest.}\end{cases}$$

(2.5) is called the $\vee$ – inferior envelope of the element $a\in C$.

$\breve{a}:C^*\to R\cup\{-\infty\},\ \breve{a}(\mu)=\inf_{v\in V}\breve{a}_v(\mu)$ is called the inferior envelope of $a\in C$.

• The notion "envelope" of an element from C was introduced with the intention to obtain a simple characterization of the sup (sub) harmonic elements.

• Proposition 2.3.: Let(C,V), be a locally convex cone and $G\subset C$, a subcone. Let $a\in C$ and $\mu\in C^*$. Then, the following affirmation we true: 1. $\hat{a}$ is sublinear;

2. $\mu(a)\le\hat{a}(\mu)$ (= if $a\in G$);

3. $a\mapsto\hat{a}(\mu)$ is continuous in the inferior topology on C.

Similar properties are true for inferior topology on C.

• Theorem 2.4.: Let (C,V) be a locally convex cone and $G\subset C$, subcone. Let $a\in C$ and $\mu\in C^*$ so that $\mu(a)<+\infty$. Then

1. $a\in Sup_G(\mu)\Leftrightarrow\mu(a)=\hat{a}(\mu)$;
2. $a\in Sub_G(\mu)\Leftrightarrow\mu(a)=\breve{a}(\mu)$.

- Corollary 2.5.: If (C,V) is a locally convex cone and $G\subset C$, a subcone so that $V\in G$ (resp. $(-V)\in G$) and $a\in C$, $\mu\in C^*$ so that $\mu(a)<+\infty$. Then:

$a\in Sup_G(\mu)\ Sub_G(\mu)\ ) \Leftrightarrow \mu(a)=\inf\{\mu(b)\mid b\in G, a\prec b\}$ (2.6)

($=\sup\{\mu(b)\mid b\in G, b\prec a\}$). (2.7)

- Corollary 2.6.: In the subcone G of a locally convex cone, the following affirmations are true:
  a) $\overline{G}=Sup_G(0^*)$
  b) $\overline{\overline{G}}=Sub_G(0^*)$

- Let $(\mu_\alpha)_{\alpha\in A},(\nu_\alpha)_{\alpha\in A}$ with $\mu_\alpha,\nu_\alpha\in C^*,(\forall)\alpha\in A$. We consider the pre-order relation: $(\mu_\alpha)_{\alpha\in A}\prec_G(\nu_\alpha)_{\alpha\in A}\Leftrightarrow(\forall)b\in G,(\mu_\alpha(b))_{\alpha\in A}\prec(\nu_\alpha(b))_{\alpha\in A}$

- Definition 2.7.: Let (C,V), a locally convex cone, $G\subset C$, subcone and $M\subset C^*$.

a) $a\in C$ is a G-superharmonic on M and we notate $a\in Sup_G(M)$ if:

1) $(\forall)\mu\in M,\mu(a)<+\infty$
2) $(\forall)(\mu_\alpha)_{\alpha\in A}\subset M$, $u-$equicontinuous and $(\forall)(\nu_\alpha)_{\alpha\in A}\subset C^*$, $u-$ equicontinuous, $(\nu_\alpha)_{\alpha\in A}\prec_G(\mu_\alpha)_{\alpha\in A}\Rightarrow(\nu_\alpha(a))_{\alpha\in A}\prec(\mu_\alpha(a))_{\alpha\in A}$.

b) $a\in C$ is called G-subharmonic on M and we notate $a\in Sub_G(M)$ if:

1) $(\forall)\mu\in M,\mu(a)<+\infty$;
2) $(\forall)(\mu_\alpha)_\alpha\subset M$, $u-$equicontinuous and $(\forall)(\nu_\alpha)_\alpha\subset C^*$, $u-$ equicontinuous, $(\mu_\alpha)_\alpha\prec_G(\nu_\alpha)_\alpha\Rightarrow(\mu_\alpha(a))_\alpha\prec(\nu_\alpha(a))_\alpha$.

- Proposition 2.8.: Let (C,V) a locally convex cone with all the element bounded and $G\subset C$, a subcone. Then: $Sup_G(C^*)=Sub_G(C^*)=\overline{G}^s=Sup_G(0^*)=Sub_G(0^*)$ (for the last two equalities: in the symmetric topology on C.

Results of KKT-type

- The notion of Korovkin – closure was introduced by Baskakov (1961) and then it was used in their researches by Krasnoselski, Lifshitz, Shaskin.

- Definition 3.1.: Let (C,V), a locally convex cone and $G\subset C$, a subcone.

$K_U(G)=\{a\in C\mid(\forall)(D,W)$, a locally convex cone, $(\forall)(T_\alpha)_\alpha$, $u-$ equicontinuous, $T_\alpha:C\to D$ linear, $(\forall)S:C\to D$, linear and $u-$ equicontinuous with $T_\alpha(b)\to S(b)$, $(\forall)b\in G\Rightarrow\ \Rightarrow T_\alpha(a)\to S(a)\ \}$ is called G-universal Korovkin cone.

$K_i(G)=\{a\in C\mid(\forall)(T_\alpha)_{\alpha\in A}$, $u-$equicontinuous, $T_\alpha:C\to C$, linear with $T_\alpha:(b)\to b$, $(\forall)b\in G\Rightarrow T_\alpha(a)\to a\}$ is called G- Korovkin cone.

- Definition 3.2.: Let (C,V), a locally convex cone and $G\subset C$, a subset. Then G is called a Korovkin system for C iff $K_I(G_0)=C$ where G_0 is the subcone generated by G.

- Theorem 3.3.: Let (C,V), a locally convex cone and $G\subset C$, a subcone. Let $M\subset C^*$ a subset with the following property: $(\forall)(D,W)$ l.c.c., $(\forall)S:C\to D$ linear and $u-$continuous $(S^*)^{-1}(M)$ is strictly separating for D. Then $A_G(M)\subset K_U(G)$.

- Corollary 3.4.: Let (C,V), a locally convex cone and $G\subset C$, a subcone. Let $M\subset C^*$ strictly separating for C. Then $A_G(M)\subset K_I(G)$.

- Proposition 3.5.: Let $\mathcal{X}$, a compact space, $G\subset C(X)$ a linear subspace, $f\in C(X)$.

The following affirmations an equivalent:

(1) $f\in K_I(G)$;

(2) $(\forall)x\in X,\ f(x)=\sup_{\varepsilon>0}\inf\{g(x)\mid g\in G, f\le g+\varepsilon\}=$
$=\inf_{\varepsilon>0}\sup\{g(x)\mid g\in G, g\le f+\varepsilon\}$

(3) $(\forall)x\in X,\ (\forall)\mu\in M^+(X),\ \mu=\varepsilon_x$ on $G\Rightarrow\mu(f)=f(x)$.

- Proposition 3.6.: Let $\mathcal{X}$, a Haussdorff compact space, $G\subset C(X)$ a subspace and $f\in C(X)$.

The followings an equivalent:

(1) $(\forall)(T_\alpha)_\alpha$, $T_\alpha:C(X)\to C(X)$ contractiveso that $T_\alpha(g)\to g$, $(\forall)g\in G\ \Rightarrow$
$\Rightarrow T_\alpha(f)\to f$.

(2) $(\forall)\ x\in X,\ f(x)=\inf\{g(x)+\|f-g\|\mid g\in G\}$.

(3) $(\forall)x\in X,\ (\forall)\mu\in M_1^+\ (X)$
$\mu(g)=g(x),\ (\forall)g\in G\Rightarrow\mu(f)=f(x)$.

- Proposition 3.7.: Let (C,V), a locally convex cone with all the elements bounded. Then $(G,\overline{V})$, subcone for $(\overline{DConv(C)},\overline{V})$ so that:
  1. C, subconefor G;
  2. $(\forall)B\in G$ is precompact in the superior topology on C;
  3. $(\forall)B_1,B_2\in G,\ \overline{co}(B_1\cup B_2)\in G$.

Let be $(T_\alpha)_\alpha$, $u-$equicontinuous, $T_\alpha:C\to C$ linear so that $\overline{T}_\alpha\downarrow_G id$ (where $\overline{T}$ is the canonique). Then, $(\forall)A\subset C$ is a convex and precompact in the superior topology $\overline{T}_\alpha(A)\downarrow A$.

- Corollary 3.8.: Let E, alocally convex space and V, a base for $\mathcal{V}(0)$.

If $U\subset Conv(E)$, subcone and $(T_\alpha)_\alpha$ equicontinuous, $T_\alpha:E\to E$ linear so that $(\forall)U\in U$, $(\forall)V\in V$, $(\exists)\alpha_0$ a.i. $U\subset T_\alpha(U)+V$, $(\forall)\alpha\ge\alpha_0$. Then, $(\forall)A\subset E$ convex and precompact, $(\forall)V\in V$, $(\exists)\alpha_1$ so that. $A\subset T_\alpha(A)+V$, $(\forall)\alpha\ge\alpha_1$.

- Propositions 3.5 and 3.6 are due to Bauer, Baskakov, Berens, Lorentz, Rubinov, Kutateladze: characterizations for $K_I(G)$ şi $K_I'(G)$.

- And analogous characterization is obtained in the same manner for function spaces because R_ε- homeomorphisms are $d-$directed operators, and $M\equiv$ the Şilov boundary for E ($S:E\to F$ linear adn positive is called R_ε- homeomorphisms on E and F iff: $f_1,f_2\in E,\ g\in F, S(f_i)\le g,\ i=1,2,(\forall)\varepsilon>0,(\exists)f_\varepsilon\in E$ so that. $f_i\le f_\varepsilon$ and $S(f_\varepsilon)\in g+\varepsilon$).

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