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ANALYTICAL AND EXPERIMENTAL DETERMINATION OF STRESSES IN THE PLANE COMPOSITE PLATES

Corneliu MOROIANU[1]
[1] Associate profesor, PhD, ‘’Mircea cel Bătrân’’Naval Academy, Constanta

Abstract: *This work presents the mathematical method for solving the differential equations by means of which we can determine the stresses in the plane composite plates used to build crafts. The results analytically determined are compared with the experimental ones. The first chapter of the work presents the differential equation system for composite materials (glass reinforced polyester resin) to determine the stress function F(x,y) and the deflection w(x,y) by which the stress condition in the composite is established. In chapter 2 they are presented the analytical resolution methods (simple and double trigonometric series), approximate analytical methods (Ritz) and finite element method (COSMOS and ALGOR programs). In chapter 3 the theoretical results are compared with the experimental ones obtained on a plate model made by five woven roving plies produced in FIROS S.A. Bucharest. The impregnation resin is NESTRAPOL 450 produced by POLICOLOR S.A. Bucharest.*

1. THE DIFFERENTIAL EQUATION SYSTEM FOR COMPOSITE MATERIAL PLATES

The fundamental researches on the composite materials (with material orthotropy) are in process of development. By applying the elements of “The elastic theory” to the composite plates normally and in median plane stressed, the differential equations of strained median surfaces for deflected plate (*0,5<w<5h*) are:

$$\begin{aligned} &E_x \frac{\partial^4 F}{\partial x^4} + 2E_{xy} \frac{\partial^4 F}{\partial x^2 \partial y^2} + E \frac{\partial^4 F}{\partial y^4} = E_x E_y \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \\ &D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = h \left[\frac{p(x,y)}{h} + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \end{aligned} \tag{1}$$

By analyzing the equation system (1) and comparing it with the equation system for isotropic plate (steel or aluminum) we note the appearance of stiffness on the two directions which changes the structure of solutions.

*Dept. of Naval Machinery and Equipment, „Mircea cel Batran” Naval Academy, Constanța.

The equation system (1) has as unknowns the stress function *F(x,y)* and the deflection *w(x,y),* which can be determined by means of the boundary conditions for various supporting ways (rigid fixing, simple or free side suspension). In the previous work, the special forms of equation system (1) have been presented, from which we are interested in particular, in the rigid plate with small deflection ($w \leq 0.2h$) of the following form:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p(x,y) \tag{2}$$

The presence of a small strain of the plate means that the ship’s shape changes very little due to the water action, the stream lines don’t change very much and the heading resistance doesn’t increase very much due to the hull ‘s strain.

2. THE SOLVING METHODS OF DIFFERENTIAL EQUATIONS

The mathematical resolution of differential equation system (1) is possible only in particular cases. So, it is necessary a careful analysis of strength structure of ship and her skin. Taking into account only the local loading, the ship’s strength structure is formed both by keelsons, girders and lines alongside and floors, frames and beams athwart wise forming a network on which the ship’s skin is fixed. I consider the plate mesh, between the stiffening members, stressed by water pressure, being rigid with a small deflection ($w \leq 0.2h$) where the sectional stresses N_x , N_y , N_{xy} don’t influence the bending. In this case, the equation system (1) under the form of (2) represents a linear differential equation with partial derivates and constant coefficients. To determine the stress and strain conditions in the plate mesh resulted from the local loading, means to find a function *w* which to check the differential equation (2) and in the same time the boundary conditions depending on the supporting pattern.

The solving methods can be divided into:

a) **Accurate methods:** when the solutions are obtained by integrating the differential equations, determining the solution of homogeneous equation and a solution for inhomogeneous equation. The most usual solutions are: by polynomials, by simple or double trigonometrically series, by hyperbolic solutions.
b) **Aproximativ analytical methods (energetically):** when the unknown function, *w,* is approximated, from energetically reasons, satisfying both the system and the supporting conditions on the contour line. They are: the orthogonally method, Ritz, Rayleigh-Ritz, Bubnov-Galerkin, Treftz, etc.
c) **Aproximativ numerical methods**: the finite element method or the finite difference method.

The numerical calculus has been carried out for a rectangular plate mesh, made of five plied laminar with its sides: a = 350 mm, b = 700 mm, h = 4.3 mm which can be used to a ship of a length of 5.5 m and a draft of d = 0.3 m. The load resulted from the water pressure is: $p = \rho g d = 3000\ N/m^2$. The measured equivalent elastic constants are: $E_1 = 1.872\ 10^4\ N/mm^2$, $E_2 = 1.826\ 10^4\ N/mm^2$, $G_{12} = 0.2637\ 10^4\ N/mm$, $\mu_{12} = \mu_{21} = 0.141$.

2.1 The analytical resolution of differential equations by double trigonometrically series

We consider the general case when the plate is of *a x b x h*, simply supported on the contour line, normally loaded with *p(x,y)* varying on both directions. The system of axes is like in Figure 1.

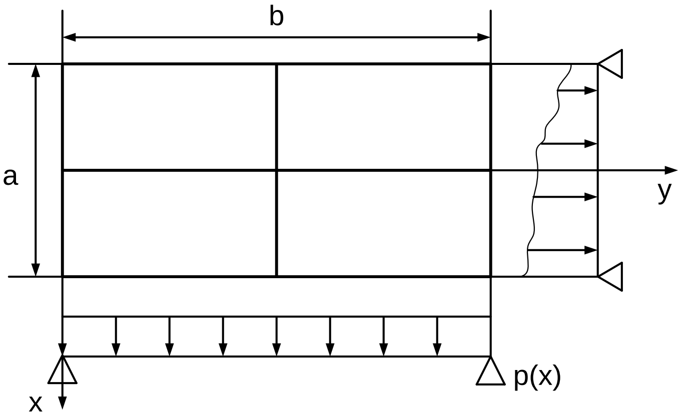

Figure 1. Plate loaded with a load distributed on the surface varying on both directions

It is developed the normal load *p(x,y)* in double Fourier's series: $p(x,y)=\sum_m\sum_n p_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

m, n = 1,2,3, (3)

The parameters p_{mn} are determined by Euler's method and become:

$$p_{mn}=\frac{4}{ab}\int_0^a\int_0^b p(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\qquad \text{m, n = 1,3,5,…}\qquad (4)$$

For the load uniformly distributed p_o the parameters p_{mn} become:

$$p_{mn}=\frac{16p_0}{\pi^2 mn}\qquad \text{m, n=1,3,5, …}\qquad (5)$$

The strain function (deflection) is also developed in Fourier's series under the form of:

$$w(x,y)=\sum_m\sum_n w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\qquad (6)$$

The coefficients w_{mn} are determined from the condition that the expression (6) to satisfy the differential equation of the plate (2) for any values *x, y* and the boundary conditions on the contour line (for the plate simply supported $w = 0$ and $M_x = M_y = 0$) and it is obtained:

$$w_{mn}=\frac{16p_0}{\pi^6 mn}\frac{1}{D_x\frac{m^4}{a^4}+2H\frac{m^2n^2}{a^2b^2}+D_y\frac{n^4}{b^4}}\qquad (7)$$

The expression of deflection (6) for the particular case when the load $p(x,y) = p_0$, the case of bottom plates of the ship, becomes:

$$w(x,y)=\frac{16p_0}{\pi^6}\sum_m\sum_n\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(D_x\frac{m^4}{a^4}+2H\frac{m^2n^2}{a^2b^2}+D_y\frac{n^4}{b^4}\right)}\qquad (8)$$

The expressions of sectional moments become:

$$M_x=\frac{16p_0}{\pi^4}\sum_m\sum_n\frac{\left(D_x\frac{m^2}{a^2}+D_1\frac{n^2}{b^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(D_x\frac{m^4}{a^4}+2H\frac{m^2n^2}{a^2b^2}+D_y\frac{n^4}{b^4}\right)}$$

$$M_y=\frac{16p_0}{\pi^4}\sum_m\sum_n\frac{\left(D_y\frac{n^2}{b^2}+D_1\frac{m^2}{a^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(D_x\frac{m^4}{a^4}+2H\frac{m^2n^2}{a^2b^2}+D_y\frac{n^4}{b^4}\right)}\qquad \text{m, n = 1,3,5,}\qquad (9)$$

$$M_{xy} = -\frac{32 p_0}{\pi^4 ab} \sum_m \sum_n \frac{D_{xy} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}}{\left(D_x \frac{m^4}{a^4} + 2H \frac{m^2 n^2}{a^2 b^2} + D_y \frac{n^4}{b^4} \right)}$$

The expressions of shearing forces become:

$$T_x = \frac{16 p_0}{\pi^3} \sum_m \sum_n \frac{\left[D_x \left(\frac{m}{a} \right)^3 + H \frac{m}{a} \left(\frac{n}{b} \right)^3 \right] \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn\left(D_x \frac{m^4}{a^4} + 2H \frac{m^2 n^2}{a^2 b^2} + D_y \frac{n^4}{b^4} \right)}$$

$$T_y = \frac{16 p_0}{\pi^3} \sum_m \sum_n \frac{\left[D_y \left(\frac{n}{b} \right)^3 + H \frac{n}{b} \left(\frac{m}{a} \right)^3 \right] \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}}{mn\left(D_x \frac{m^4}{a^4} + 2H \frac{m^2 n^2}{a^2 b^2} + D_y \frac{n^4}{b^4} \right)} \qquad (10)$$

The maximum deflection is produced at the middle of the plate, that is, in the coordinate point x = *a/2* and *y = b/2*, and in this case the relation becomes:

$$w_{\max} = \frac{16 p_0}{\pi^6} \sum_m \sum_n \frac{(-1)^{\frac{m+n}{2}-1}}{mn\left(D_x \frac{m^4}{a^4} + 2H \frac{m^2 n^2}{a^2 b^2} + D_y \frac{n^4}{b^4} \right)} \text{; m,n = 1,3,5,...} \qquad (11)$$

a) The plate simply supported loaded with *p(x,y)* = p_0 = ct.
b) The diagram of stresses in the rectangular plate simply supported.

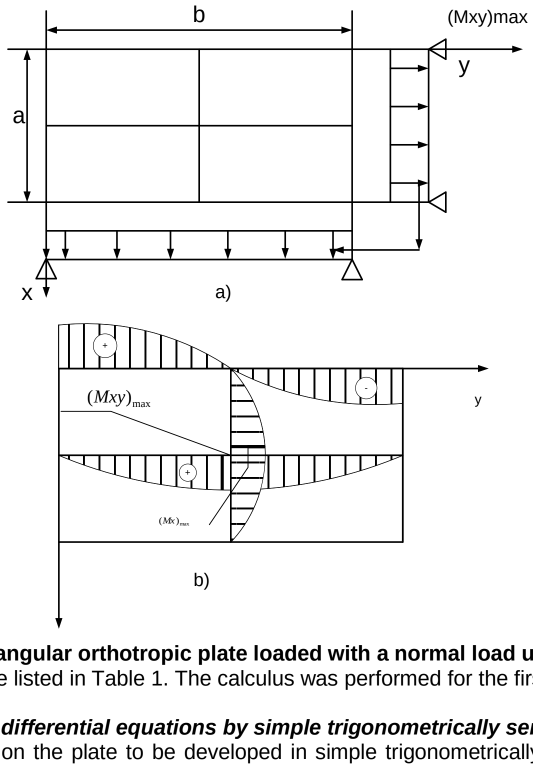

Figure 2. The rectangular orthotropic plate loaded with a normal load uniformly distributed

The numerical results are listed in Table 1. The calculus was performed for the first three terms.

2.2 The analytical resolution of differential equations by simple trigonometrically series

For the normal load (*p*) on the plate to be developed in simple trigonometrically series on the chosen direction, for instance, on *0x,* it must obey two conditions (Figure 3):

a) The load *p* must not vary on *0y*;
b) The function *p(x)* must obey the Diricht’s conditions.
c) To solve the differential equation (2) it is considered the expression of the deflection under the form of:

d) $w_1 = \sum_k \mathrm{Y}_k \sin\frac{k\pi x}{a}$ or $w_2 = \sum_k \mathrm{X}_k \sin\frac{k\pi y}{b}$ (12)

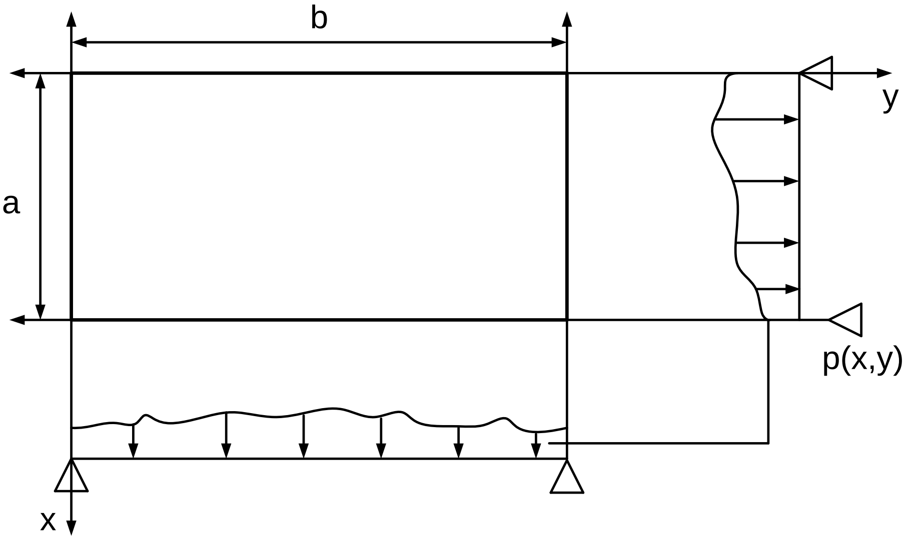

Figure 3.The plate simply supported loaded with *p(x)* varying on the direction

For the plate in the Figure 3 we take the form:

$$w(x,y)=\sum_k Y_k(y)\sin\frac{k\pi x}{a} \tag{13}$$

where:

Y_k – unknown factors formed of the arbitrary functions only of *y*.

The load *p(x)* is developed in simple Fourier's series, as an odd function, varying only on *Ox* axis.

$$p(x)=\sum_k p_k \sin\frac{k\pi x}{a} \qquad k = 1,2,3, \ldots \tag{14}$$

where:

$$p_k=\frac{2}{a}\int_0^a p(x)\sin\frac{k\pi x}{a}dx=\frac{2p}{k\pi}(1-\cos k\pi) \tag{15}$$

The function Y_k is determined from the condition that the deflection *w(x,y)* to satisfy both the boundary conditions and the plate equation.

The biquadrate differential equation, linear, inhomogeneous, with constant coefficients, is:

$$Y_k^{IV}-\frac{2H}{D_y}\left(\frac{k\pi}{a}\right)^2 Y_k^{''}+\frac{D_x}{D_y}\left(\frac{k\pi}{a}\right)^4 Y_k=\frac{p_k}{D_y} \tag{16}$$

Taking: $m^2=\frac{k^2\pi^2}{a^2}\frac{H}{D_y}$ şi $n^4=\frac{k^4\pi^4}{a^4}\frac{D_x}{D_y}$, the characteristic equation is:

$$r^4-2m^2r^2+n^4=0 \tag{17}$$

For the anisotropic plates, it appears the inequality $H < D_x$, that is why all the roots will be complex and under the form of:

$$r_{1,2,3,4}=\pm\alpha\pm\beta\cdot i \tag{18}$$

where:

$$\alpha=\sqrt{\frac{n^2+m^2}{2}} \text{ and } \beta=\sqrt{\frac{n^2-m^2}{2}}.$$

The general solution of differential equation (16) can be written as:

$$Y_k(y)=A_k ch\alpha y+B_k sh\alpha y+C_k \alpha y ch\alpha y+D_k \alpha y sh\alpha y+Y_{kp} \tag{19}$$

where:

$Y_{kp}=\frac{2pa^4}{D_y k^5\pi^5}(1-\cos k\pi)$ - is a particular solution of inhomogeneous differential equation.

The deflection equation becomes:

$$w(x,y)=\sum_k\left[A_k ch\alpha y+B_k sh\alpha y+C_k\alpha y ch\alpha y+D_k\alpha y sh\alpha y+\frac{2pa^4}{k^5\pi^5}(1-\cos k\pi)\right]$$

k = 1,3,5... (20)

If the plate is symmetrical about Ox axis, the bent / deflected surface is symmetrical about Ox axis and also the load is symmetrical, Y_k must be an even function, it results that $B_k = C_k = 0$. The general solution is:

$$Y_k(y)=A_k ch\alpha y+D_k\alpha y sh\alpha y+\frac{2pa^4}{k^5\pi^5}(1-\cos k\pi) \qquad (21)$$

The integration constants are determined from the boundary conditions (of contour line) for the simply supported plate and they are:

$$A_k=-\frac{2pa^4}{D_y k^5\pi^5}(1-\cos k\pi)-\frac{\left(\alpha_k^2-\beta_k^2\right)sh\frac{\alpha_k b}{2}+2\alpha_k\beta_k ch\frac{\alpha_k b}{2}}{2\alpha_k\beta_k\left(ch^2\frac{\alpha_k b}{2}+sh^2\frac{\alpha_k b}{2}\right)}=f_k\frac{C_{2k}^*}{k\pi}$$

$$D_k=\frac{2pa^4}{D_y k^5\pi^5}(1-\cos k\pi)-\frac{\left(\alpha_k^2-\beta_k^2\right)ch\frac{\alpha_k b}{2}+2\alpha_k\beta_k sh\frac{\alpha_k b}{2}}{2\alpha_k\beta_k\left(ch^2\frac{\alpha_k b}{2}+sh^2\frac{\alpha_k b}{2}\right)}=f_k\frac{C_{4k}^*}{k\pi}$$

(22)

The elastic surface of the plate is:

$$w(x,y)=\sum_k\frac{2pa^4}{D_y k^5\pi^5}\left[1-\frac{\left(\alpha_k-\beta_k\right)sh\frac{\alpha_k b}{2}+2\alpha_k\beta_k ch\frac{\alpha_k b}{2}}{2\alpha_k\beta_k\left(ch^2\frac{\alpha_k b}{2}+sh^2\frac{\alpha_k b}{2}\right)}ch\alpha y+\right.$$

(23)

$$\left.+\frac{\left(\alpha_k^2-\beta_k^2\right)ch\frac{\alpha_k b}{2}-2\alpha_k\beta_k sh\frac{\alpha_k b}{2}}{2\alpha_k\beta_k\left(ch^2\frac{\alpha_k b}{2}+sh^2\frac{\alpha_k b}{2}\right)}sh\alpha y\right](1-\cos k\pi)\sin\frac{k\pi x}{a}$$

The bending moments (M_x, M_y) and the twisting moment (M_{xy}) are calculated and the numerical results are listed in Table 1.

2.3 Resolution of differential equation by Ritz method

By neglecting the influence of shearing forces, the expression of strain potential energy of an orthotropic plate during the bending, taking into account the function (2), is:

$$U_i=\frac{1}{2}\iint\left\{\left[D_x\frac{\partial^2 w}{\partial x^2}+D_y\frac{\partial^2 w}{\partial y^2}\right]^2-\left[D_1^2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(2D_{xy}\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}dxdy$$

(24)

The potential energy of external forces has the expression:

$$U_e=-\iint p(x,y)w(x,y)dxdy \qquad (25)$$

The expression of total potential energy is:

$$\Pi=U_i+U_e \qquad (26)$$

The deflection is:

$$w(x,y)=c_1w_1(x,y)+c_2w_2(x,y)+\ldots+c_nw_n(x,y) \qquad (27)$$

where:

w_i - the arbitrary functions which verify the boundary conditions on the plate contour;
c_i - the integration constants.
The minimum condition leads to:

$$\frac{\partial\pi}{\partial c_1}=0;\ \frac{\partial\pi}{\partial c_2};\ldots;\frac{\partial\pi}{\partial c_n}=0 \qquad (28)$$

from which a linear system of polynomial equations results by means of which the integration constants c_i are determined and the problem is solved.
For the plate mesh considered, we take the following function for the deflection:

$$w = c_1w_1 + c_2w_2 \qquad (29)$$

where:

$$w_1 = \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$w_2 = \sin\frac{3\pi x}{a}\sin\frac{3\pi y}{b}$$

- are functions which obey the boundary conditions.

The strain potential energy is:

$$U_i = \frac{1}{2}\left\{c_1^2\left(D_x\frac{\pi^2}{a^2} + D_y\frac{\pi^2}{b^2}\right)^2\frac{ab}{4} + c_2\left[D_x\left(\frac{3\pi}{a}\right)^2 + D_y\left(\frac{3\pi}{b}\right)^2\right]^2\frac{ab}{4}\right\} \quad (30)$$

The potential energy of external forces is:

$$U_e = -p\left(\frac{4ab}{\pi^2}c_1 + \frac{4ab}{9\pi^2}c_2\right) \quad (31)$$

Taking the minimum condition (28), we have:

$$\begin{cases} \frac{1}{2}\cdot 2c_1\left(D_x\frac{\pi^2}{a^2} + D_y\frac{\pi^2}{b^2}\right)\frac{ab}{4} - \frac{4ab}{\pi^2}p = 0 \\ \frac{1}{2}\cdot 2c_2\left[D_x\left(\frac{3\pi}{a}\right)^2 + D_y\left(\frac{3\pi}{b}\right)^2\right]^2 - \frac{4ab}{9\pi^2}p = 0 \end{cases} \quad (32)$$

From the system (32) we obtain the values of the constants:

$$c_1 = \frac{16p}{\pi^6\left(D_x\frac{1}{a^4} + D_y\frac{1}{b^4}\right)}$$

$$c_2 = \frac{16p}{9\pi^6\left[D_x\left(\frac{3}{a}\right)^2 + D_y\left(\frac{3}{b}\right)^2\right]^2} \quad (33)$$

We easily see that these coefficients are in fact, the coefficients of double trigonometrically series.
The deflection will have the following form:

$$w = \frac{16p}{\pi^6}\left\{\frac{\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{\frac{D_x}{a^4} + \frac{D_y}{b^4}} + \frac{\sin\frac{3\pi x}{a}\sin\frac{3\pi y}{b}}{9\left[D_x\left(\frac{3}{a}\right)^2 + D_y\left(\frac{3}{b}\right)^2\right]^2}\right\} \quad (34)$$

The moments and the shearing forces can be calculated and the numerical results are listed in Table 1.

2.4 Resolution of differential equation by M.E.F (COSMOS M and ALGOR).

Due to the geometrical and loading symmetry of the plate simply supported with fixed ends, it has been studied ¼ of the plate discussed into 2450 quadratic finite elements with the side of 5 x 5 mm forming a network with 2556 mesh points. The study was performed by two different programs (COSMOS M and ALGOR).

To study and compare the results of the two cases, we keep the same loading, discussion and elements on the symmetry axis. Regarding the parameters, it was studied the deflection, turning, bending and twisting moments, unit and equivalent stresses on each element. The results are listed in Table 1.

3. THE EXPERIMENTAL RESULTS

To check the value of maximum deflection obtained by the theoretical methods mentioned above, we built a device by means of which we measured the maximum deflection in the middle of the plate. The device is formed of two rigid angle bar frames by means of which we performed the fixing and with only one frame we made the support on the sides. The loading was made with fine, dry sand with a density of ρ = 1.3 kg/dm^2 . The thickness of sand layer was calculated from the condition of loading with a load uniformly distributed *p* = 3000N/m^2 . The deflection was measured in the middle of the plate by a comparator. The comparation between the calculated values and the measured ones for the five plied laminar is shown in Table 1.

Table 1

Method of determination	Maximum analyzed values					
	w_{max} [mm]	M_x [mm/mm]	My[mm/mm]	M_{xy} [mm/mm]	σ_x[mm/mm]	Σ_y [mm/mm]
The plate simply supported						
Experimental	4,2	43,601	11,91	-6,91	14,15	3,86
Double trigonometrical series	4,383	-	-	-	-	-
Simple trigonometrical series	1,413	44,157	16,31	-5,44	-	-
Ritz method	4,18	44,8	13,2	-8,31	14,57	4,278
MEF (COSMOS M program)	4,424	-	-	-	-	-
MEF (ALGOR program)	4,205	-	-	-	-	-
The fixed plate						
Experimental	1,19	-	-	-	-	-
MEF (COSMOS M program)	0,961	16	3,44	-	5,91	1,12

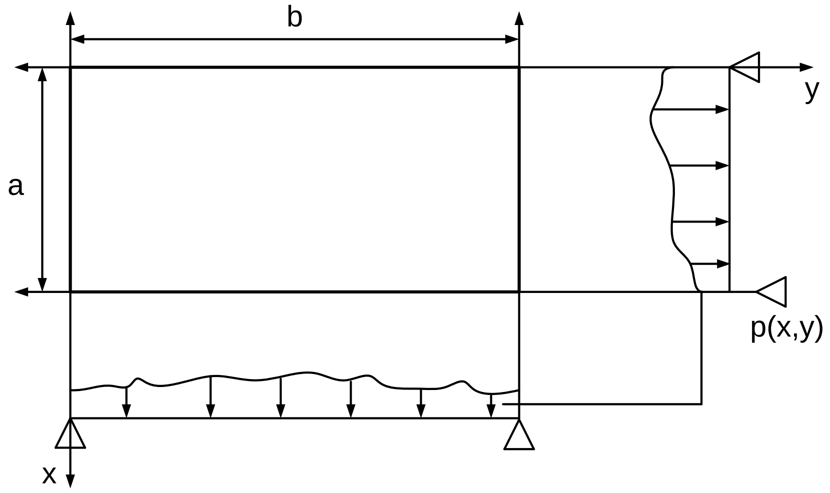

Figure 4. The plate studied with symmetry planes and coordinate system of axes. The hatched part was studied

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