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FUNCTIONS AND CHANGES OF VARIABLES

Gheorghe DOGARU[1]
Camelia CIOBANU[2]
Gheorghe SAMOILESCU[3]
[1] Senior lecturer PhD, „Mircea cel Batran" Naval Academy, Constanta
[2] Associate professor PhD, „Mircea cel Batran" Naval Academy, Constanta
[3] Professor PhD, „Mircea cel Batran" Naval Academy, Constanta

Abstract: *In this paper, we outline how to simplify the differential expression that arise in various mathematical models used by specialized technical areas, by introducing differential notation that simplify formulas.*
Key-words: *partial derivatives, regular punctual transformation, system equations*

1. INTRODUCTION

In practice are many physical phenomena which allow mathematical modeling in which appear relationships between a function of several variables and its partial derivatives. This means that in the mathematical model of these phenomena we can find partial differential equations of various orders and forms.

Most often the state of a physical environment at a point M(x,y,z) at a time t is described by a function u(x,y,z) which accept continues partial derivatives of first and second order.

For exemple:

For vibratory phenomena are obtained differential equations of the form:

$A(x)\cdot\frac{\partial^2 u}{\partial t^2}-B\cdot\frac{\partial^2 u}{\partial x^2}=F(x,t)$ (Equation of the vibrant chord)

$\frac{\partial^2 u}{\partial t^2}-B(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})=F(x,y,t)$ (Equation of the vibrating membrane)

For wave events Schrodinger's equation is obtained: $i\cdot h\cdot\frac{\partial\psi}{\partial t}+\frac{h^2}{2m}\Delta\psi=V\psi$,

where m is the mass of a particle in motion under the action of a external force field with potential *V(x,y,z)*, *ψ(x,y,z,t)* is the wave function, h is Planck's constant and $i=\sqrt{-1}$.

For the propagation of the heat in homogeneous medium is obtained the differential equation:

$\frac{\partial u}{\partial t}-a\Delta u=q(x,y,z,t)\left(\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$, etc.

2. UTILIZATION OF FUNCTIONS AND CHANGES OF VARIABLES

There are many practical situations where point is situated in domains $D\subset\mathbb{R}^3$ of different shapes which leading to differential equations more complicated than those shown in the introduction. If z=z(x,y), then the domain D is an area from $\mathbb{R}^2$.

It is necessary to use some special transformations, $T:\begin{cases}u=f(x,y,z)\\ v=g(x,y,z)\\ w=h(x,y,z)\end{cases},(x,y,z)\in D\subset\mathbb{R}^3,$ to obtaining the solutions for such differential equations like those from above. In this case the domain D it will be transformed in D'by equation *w=w(u,v)*which express a new surface.

For this reason it is necessary to express the partial derivatives of various orders of the function which defines the old domain, function of partial derivatives for the function which defines the new field or vice versa.

For this, further we will introduce some expressions containing partial derivatives and with these, it will be easy to obtain symmetric formulas (easy to remember) between the derivatives for the old and the new function.

To simplify writing the following notations will be used:

$F_x=f'_x+f'_z\cdot z'_x;\ F_y=f'_y+f'_z\cdot z'_y$

Replacing f' with g' respectively h' we obtain G_x, G_y, H_x, H_y, ie

$$G_x=g'_x+g'_z\cdot z'_x;\ G_y=g'_y+g'_z\cdot z'_y;\ H_x=h'_x+h'_z\cdot z'_x;\ H_y=h'_y+h'_z\cdot z'_y$$

Also to simplify the writing we introduce the following notation:

$$F_{x^2}=f''_{x^2}+f''_{z^2}\cdot z''_{x^2}+2f''_{xz}\cdot z'_x;\ F_{y^2}=f''_{y^2}+f''_{z^2}\cdot z''_{y^2}+2f_{yz}\cdot z'_y$$

$$F_{z^2}=f''_{z^2}\cdot z''_{xy}+f''_{xz}\cdot z'_y+f''_{yz}\cdot z'_x+f''_{xy};$$

If we replace f " with g " respectively h " we will obtain $G_{x^2}, G_{Y^2}, G_{Z^2}$; $H_{x^2}, H_{Y^2}, H_{Z^2}$ meaning

$$G_{x^2}=g''_{x^2}+g''_{z^2}\cdot z''_{x^2}+2g''_{xz}\cdot z'_x;\quad G_{y^2}=g''_{y^2}+g''_{z^2}\cdot z''_{y^2}+2g_{yz}\cdot z'_y;$$

$$G_{z^2}=g''_{z^2}\cdot z''_{xy}+g''_{xz}\cdot z'_y+g''_{yz}\cdot z'_x+g''_{xy}$$

$$H_{x^2}=h''_{x^2}+h''_{z^2}\cdot z''_{x^2}+2h''_{xz}\cdot z'_x;\ H_{y^2}=h''_{y^2}+h''_{z^2}\cdot z''_{y^2}+2h_{yz}\cdot z'_y$$

$$H_{z^2}=h''_{z^2}\cdot z''_{xy}+h''_{xz}\cdot z'_y+h''_{yz}\cdot z'_x+h''_{xy}$$

If it is considered the surface (S):z=z(x,y) where (x,y) $\in D \subset \mathbb{R}^2$ and the regular punctual transformation *T:* $\begin{cases} u = f(x,y,z) \\ v = g(x,y,z) \\ w = h(x,y,z) \end{cases}$, (x,y,z)$\in A \subset \mathbb{R}^3$, it will be obtain the surface $(\Sigma): w = w(u,v)$, where *w=T(z).*

To study the surface properties (Σ) with respect the surface's properties(S) it must be experienced the derivatives $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}, \frac{\partial^2 w}{\partial u^2}, \frac{\partial^2 w}{\partial v^2}, \frac{\partial^2 w}{\partial u \partial v}$, function by the derivatives: $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$.

Proposition 1. Let $z = z(x,y)\ \ (x,y) \in D \subset \mathbb{R}^2$ a function which admits the first order derivative, continuous on D and the regular punctual transformation *T:*$\begin{cases} u = f(x,y,z) \\ v = g(x,y,z) \\ w = h(x,y,z) \end{cases}$, (x,y,z)$\in A \subset \mathbb{R}^3$. If it is considered the function *w=w(u,v)* where *w=T(z)*, then $\frac{\partial w}{\partial u} = \frac{\begin{vmatrix} H_x & G_x \\ H_y & G_y \end{vmatrix}}{\begin{vmatrix} F_x & G_x \\ F_y & G_y \end{vmatrix}}\ si\ \frac{\partial w}{\partial v} = \frac{\begin{vmatrix} F_x & H_x \\ F_y & H_y \end{vmatrix}}{\begin{vmatrix} F_x & G_x \\ F_y & G_y \end{vmatrix}}$.

Proof. We know that w=w(u,v) is differentiable and $dw = \frac{\partial w}{\partial u} du + \frac{\partial w}{\partial v} dv$ (1). On the other hand $u = f(x,y,z)$ and $v = g(x,y,z)$ are differentiable too and

$$\begin{cases} du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \\ dv = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz \end{cases} (2).$$

Also $dw = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy + \frac{\partial h}{\partial z} dz (3)$, where $z = z(x,y)$ is a differentiable function and $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy\ (4)$. Now from (2) and (4) it was obtained:

$$\begin{cases} du = F_x dx + F_y dy \\ dv = G_x dx + G_y dy \end{cases} (5).$$

From (3) and (4) it was obtained (6)$dw = H_x dx + H_y dy$, and from (1) and (5) it was obtained

$$dw = \left(\frac{\partial w}{\partial u} \cdot F_x + \frac{\partial w}{\partial v} \cdot G_x\right) dx + \left(\frac{\partial w}{\partial u} \cdot F_y + \frac{\partial w}{\partial v} \cdot G_y\right) dy\ (7).$$

Now from (6) and (7) by identification we obtain the system

$$\begin{cases} \frac{\partial w}{\partial u} \cdot F_x + \frac{\partial w}{\partial v} \cdot G_x = H_x \\ \frac{\partial w}{\partial u} \cdot F_y + \frac{\partial w}{\partial v} \cdot G_y = H_y \end{cases} (8).$$

This system represents one Crammer system equations by unknowns $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ where the determinant of the system is:

$\Delta = \begin{vmatrix} F_x & G_x \\ F_y & G_y \end{vmatrix}$ and $\Delta_{\frac{\partial w}{\partial u}} = \begin{vmatrix} H_x & G_x \\ H_y & G_y \end{vmatrix}$, $\Delta_{\frac{\partial w}{\partial v}} = \begin{vmatrix} F_x & H_x \\ F_y & H_y \end{vmatrix}$. Using Crammer's rule we obtain

$$\frac{\partial w}{\partial u} = \frac{\Delta_{\frac{\partial w}{\partial u}}}{\Delta} = \frac{\begin{vmatrix} H_x & G_x \\ H_y & G_y \end{vmatrix}}{\begin{vmatrix} F_x & G_x \\ F_y & G_y \end{vmatrix}} \text{and} \frac{\partial w}{\partial v} = \frac{\Delta_{\frac{\partial w}{\partial v}}}{\Delta} = \frac{\begin{vmatrix} F_x & H_x \\ F_y & H_y \end{vmatrix}}{\begin{vmatrix} F_x & G_x \\ F_y & G_y \end{vmatrix}}$$

Observation: a) Proposition 1 represents a change by function and variable for functions by two variables;
b) If considered a suitable punctual transformation, T, it can turn the form of some mathematical expressions which contain partial derivatives into appropriate forms necessary to solve various problems.

Exemple 1 We consider that will be interesting what could become the relation $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$, $z = z(x,y)$ in the new function $w = w(u,v)$ where $w = T(z)$ if it was used the regular punctual transformation *T*: $\begin{cases} u = x + z \\ v = y + z \\ w = x + y \end{cases}$

Considering the Proposition 1 we can write that:

$$\begin{cases} \dfrac{\partial w}{\partial u} = \dfrac{\begin{vmatrix} H_x & G_x \\ H_y & G_y \end{vmatrix}}{\begin{vmatrix} F_x & G_x \\ F_y & G_y \end{vmatrix}}; \\ \dfrac{\partial w}{\partial v} = \dfrac{\begin{vmatrix} F_x & H_x \\ F_y & H_y \end{vmatrix}}{\begin{vmatrix} F_x & G_x \\ F_y & G_y \end{vmatrix}}; \end{cases}$$

$F_x = f_x' + f_z' \cdot z_x' = 1 + z_x'$; $F_y = f_y' + f_z' \cdot z_y' = z_y'$;
$(f(x,y,z) = x + z)$;
$G_x = g_x' + g_z' \cdot z_x' = z_x'$; $G_y = g_y' + g_z' \cdot z_y' = 1 + z_y'$;
$(g(x,y,z) = y + z)$;
$H_x = h_x' + h_z' \cdot z_x' = 1$; $H_y = h_y' + h_z' \cdot z_y' = 1$;
$(h(x,y,z) = x + y)$;

so,

$$\begin{vmatrix} F_x & G_x \\ F_y & G_y \end{vmatrix} = \begin{vmatrix} 1 + z_x' & z_x' \\ z_y' & 1 + z_y' \end{vmatrix} = 1 + z_x' + z_y'$$

$$\begin{vmatrix} H_x & G_x \\ H_y & G_y \end{vmatrix} = \begin{vmatrix} 1 & z_x' \\ 1 & 1 + z_y' \end{vmatrix} = 1 - z_x' + z_y'$$

$$\begin{vmatrix} F_x & H_x \\ F_y & H_y \end{vmatrix} = \begin{vmatrix} 1 + z_x' & 1 \\ z_y' & 1 \end{vmatrix} = 1 + z_x' - z_y'$$

Therefore we obtain the following linear system in the unknowns z_x' and z_y':

$$\begin{cases} \dfrac{\partial w}{\partial u} = \dfrac{1 - z_x' + z_y'}{1 + z_x' + z_y'} \\ \dfrac{\partial w}{\partial v} = \dfrac{1 + z_x' - z_y'}{1 + z_x' + z_y'} \end{cases} \Rightarrow \begin{cases} \left(\dfrac{\partial w}{\partial u} + 1\right) z_x' + \left(\dfrac{\partial w}{\partial u} - 1\right) z_y' = 1 - \dfrac{\partial w}{\partial u} \\ \left(\dfrac{\partial w}{\partial v} - 1\right) z_x' + \left(\dfrac{\partial w}{\partial v} + 1\right) z_y' = 1 - \dfrac{\partial w}{\partial v} \end{cases}$$

This system could is solved using Crammer's rule, where:

$$\Delta = \begin{vmatrix} \dfrac{\partial w}{\partial u} + 1 & \dfrac{\partial w}{\partial u} - 1 \\ \dfrac{\partial w}{\partial v} - 1 & \dfrac{\partial w}{\partial v} + 1 \end{vmatrix} = 2\left(\dfrac{\partial w}{\partial u} + \dfrac{\partial w}{\partial v}\right)$$

and $\Delta_{z_x'} = \begin{vmatrix} 1 - \frac{\partial w}{\partial u} & \frac{\partial w}{\partial u} - 1 \\ 1 - \frac{\partial w}{\partial v} & \frac{\partial w}{\partial v} + 1 \end{vmatrix} = 2\left(1 - \frac{\partial w}{\partial u}\right)$.

Therefore we obtain $z_x' = \frac{1 - \frac{\partial w}{\partial u}}{\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}}$ (1) and analogous $z_y' = \frac{1 - \frac{\partial w}{\partial v}}{\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}}$ (2).

Given the transformation T is obtained the linear system in the unknowns *x, y, z.*

$$\begin{cases} x + 0 \cdot y + z = u \\ 0 \cdot x + y + z = v \\ x + y + 0 \cdot z = w \end{cases} \Rightarrow x = \frac{\Delta_x}{\Delta};\ y = \frac{\Delta_y}{\Delta}; z = \frac{\Delta_z}{\Delta};$$

$$\Delta = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1 - 1 = -2;\quad \Delta_x = \begin{vmatrix} u & 0 & 1 \\ v & 1 & 1 \\ w & 1 & 0 \end{vmatrix} = v - w - u;$$

$$\Delta_y = \begin{vmatrix} 1 & u & 1 \\ 0 & v & 1 \\ 1 & w & 0 \end{vmatrix} = u - v - w;\ \Delta_z = \begin{vmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 1 & 1 & w \end{vmatrix} = w - v - u;$$

So we obtain: $x = \frac{u - v + w}{2},\ y = \frac{-u + v + w}{2},\ z = \frac{u + v - w}{2}$ (3).

Given (1), (2), (3) the stated relationship is

$$(u-v+w)\left(1-\frac{\partial w}{\partial u}\right)+(-u+v+w)\left(1-\frac{\partial w}{\partial v}\right)=(u+v-w)\left(\frac{\partial w}{\partial u}-\mathbf{1}\right),$$

which means: $u\cdot\frac{\partial w}{\partial u}+v\cdot\frac{\partial w}{\partial v}=w$.

Note that the punctual chosen transformation does not change the form of given relationship.

Exemple 2. Another problem might be determining the form of the equation: $y\cdot\frac{\partial z}{\partial x}+\ z\cdot\frac{\partial z}{\partial y}=x$, $z=z(x,y)$if we consider the new function $w=w(u,v)$ where $w=T(z)$ and if it was used the regular punctual transformation $T:\begin{cases} u=\ x-y+z\\ v=\ x+y-z\\ w=-x+y+z\end{cases}$.

Taking the example above algorithm we get

$x=\frac{1}{2}(u+v)y=\frac{1}{2}(v=w),z=\frac{1}{2}(u+w)$and $z_x^{'}=\frac{-1-\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}}{\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}-1}$,

$z_y^{'}=\frac{1+\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}}{\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}-1}$. Substituting these expressions in the the above equation, this becomes $v\cdot\frac{\partial w}{\partial u}+\ w\cdot\frac{\partial w}{\partial v}=u$.

It is obvious that this example shows that the form of chosen equation does not change when apply a such punctual transformation.

Proposition 2. Let consider $=z(x,y),\ (x,y)\ \in D\subset\mathbb{R}^2$, a function that accepts partial derivatives continuous of first and second order on $\mathbb{R}^2$ and the regular punctual transformation $T:\begin{cases} u=f(x,y,z)\\ v=g(x,y,z),\\ w=h(x,y,z)\end{cases}$

(x,y,z)$\in A\subset\mathbb{R}^3$, and the function $w=w(u,v)$ where $w=T(z)$, then

$$\frac{\partial^2 w}{\partial u^2}=\frac{\begin{vmatrix} H_{x^2} & G_{x^2} & F_xG_x\\ H_{y^2} & G_{y^2} & F_yG_y\\ H_{z^2} & G_{z^2} & F_yG_x\end{vmatrix}}{\begin{vmatrix} F_{x^2} & G_{x^2} & F_xG_y\\ F_{y^2} & G_{y^2} & F_yG_y\\ F_{z^2} & G_{z^2} & F_yG_x\end{vmatrix}},\frac{\partial^2 w}{\partial v^2}=\frac{\begin{vmatrix} F_{x^2} & H_{x^2} & F_xG_x\\ F_{y^2} & H_{y^2} & F_yG_y\\ F_{z^2} & H_{z^2} & F_yG_x\end{vmatrix}}{\begin{vmatrix} F_{x^2} & G_{x^2} & F_xG_x\\ F_{y^2} & G_{y^2} & F_yG_y\\ F_{z^2} & G_{z^2} & F_yG_x\end{vmatrix}}$$

$$\text{and}\frac{\partial^2 w}{\partial u\partial v}=\frac{\begin{vmatrix} F_{x^2} & G_{x^2} & H_{x^2}\\ F_{y^2} & G_{y^2} & H_{y^2}\\ F_{z^2} & G_{z^2} & H_{z^2}\end{vmatrix}}{\begin{vmatrix} F_{x^2} & G_{x^2} & F_xG_x\\ F_{y^2} & G_{y^2} & F_yG_y\\ F_{z^2} & G_{z^2} & F_yG_x\end{vmatrix}}.$$

Proof. Because functions

$w=w(u,v);u=f(x,y,z),v=g(x,y,z),w=h(x,y,z)$

have differentials of second order can be write that:

$$d^2w=\frac{\partial^2 w}{\partial u^2}du^2+\frac{\partial^2 w}{\partial v^2}dv^2+2\frac{\partial^2 w}{\partial u\partial v}dudv\ \ (1);$$

$$\begin{cases} d^2u=\frac{\partial^2 f}{\partial x^2}dx^2+\frac{\partial^2 f}{\partial y^2}dy^2+\frac{\partial^2 f}{\partial z^2}dz^2+2\left(\frac{\partial^2 f}{\partial x\partial y}dxdy+\frac{\partial^2 f}{\partial x\partial z}dxdz+\frac{\partial^2 f}{\partial y\partial z}dydz\right)\\ d^2v=\frac{\partial^2 g}{\partial x^2}dx^2+\frac{\partial^2 g}{\partial y^2}dy^2+\frac{\partial^2 g}{\partial z^2}dz^2+2\left(\frac{\partial^2 g}{\partial x\partial y}dxdy+\frac{\partial^2 g}{\partial x\partial z}dxdz+\frac{\partial^2 g}{\partial y\partial z}dydz\right)\end{cases}(2)$$

$$d^2w=\frac{\partial^2 h}{\partial x^2}dx^2+\frac{\partial^2 h}{\partial y^2}dy^2+\frac{\partial^2 h}{\partial z^2}dz^2+2\left(\frac{\partial^2 h}{\partial x\partial y}dxdy+\frac{\partial^2 h}{\partial x\partial z}dxdz+\frac{\partial^2 h}{\partial y\partial z}dydz\right)(3)$$

The function $z=z(x,y)$ accept differentials of second order:

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+\frac{\partial^2 z}{\partial y^2}dy^2+2\frac{\partial^2 z}{\partial x\partial y}dxdy\ (4)$$

First order differentials of function

$z=z(x,y),u=f(x,y,z)siv=g(x,y,z)$ are:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad (5)$$
$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \quad (6)$$
$$dv = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz \quad (7)$$

Substituting (4) and (5) in (6) and (3) it is obtaining

$$d^2u = F_{x^2}dx^2 + F_{y^2}dy^2 + 2F_{z^2}dxdy \quad (8)$$
$$d^2v = G_{x^2}dx^2 + G_{y^2}dy^2 + 2G_{z^2}dxdy \quad (9)$$
$$d^2w = H_{x^2}dx^2 + H_{y^2}dy^2 + 2H_{z^2}dxdy \quad (10)$$

Multiplying term by term of the equalities (6) and (7) is obtaining:

$$dudv = F_x \cdot G_x dx^2 + F_y \cdot G_y dy^2 + (F_x \cdot G_y + F_y \cdot G_x)dxdy \quad (11)$$

Substituting (8), (9) and (11) in (1) it is obtaining:

$$\begin{aligned} d^2w = &\left(F_{x^2} \cdot \frac{\partial^2 w}{\partial u^2} + G_{x^2} \cdot \frac{\partial^2 w}{\partial v^2} + F_x \cdot G_x \cdot \frac{\partial^2 w}{\partial u \partial v}\right)dx^2 \\ &+ \left(F_{y^2} \cdot \frac{\partial^2 w}{\partial u^2} + G_{y^2} \cdot \frac{\partial^2 w}{\partial v^2} + F_y \cdot G_y \cdot \frac{\partial^2 w}{\partial u \partial v}\right)dy^2 \\ &+ 2\left(F_{z^2} \cdot \frac{\partial^2 w}{\partial u^2} + G_{z^2} \cdot \frac{\partial^2 w}{\partial v^2} + F_y \cdot G_x \cdot \frac{\partial^2 w}{\partial u \partial v}\right)dxdy \quad (12). \end{aligned}$$

Taking into account (10) and (12), by identification it is obtaining the system:

$$\begin{cases} F_{x^2} \cdot \dfrac{\partial^2 w}{\partial u^2} + G_{x^2} \cdot \dfrac{\partial^2 w}{\partial v^2} + F_x \cdot G_x \cdot \dfrac{\partial^2 w}{\partial u \partial v} = H_{x^2} \\ F_{y^2} \cdot \dfrac{\partial^2 w}{\partial u^2} + G_{y^2} \cdot \dfrac{\partial^2 w}{\partial v^2} + F_y \cdot G_y \cdot \dfrac{\partial^2 w}{\partial u \partial v} = H_{y^2} \\ F_{z^2} \cdot \dfrac{\partial^2 w}{\partial u^2} + G_{z^2} \cdot \dfrac{\partial^2 w}{\partial v^2} + F_y \cdot G_x \cdot \dfrac{\partial^2 w}{\partial u \partial v} = H_{z^2} \end{cases} \quad (13)$$

The system (13) is a linear system in the unknowns $\frac{\partial^2 w}{\partial u^2}$, $\frac{\partial^2 w}{\partial v^2}$, $\frac{\partial^2 w}{\partial u \partial v}$ and solving it by Crammer's rule we obtain equalities by Proposition 2.

Observation. Theoretically, Proposition 2 can be generalized to partial differential by order three or higher. In fact the calculations are very long and it is very difficult to solve such a problem.

Example 3. Let consider functions $z = z(x,y)$ and $w = w(u,v)$ such that $w = T(z)$, where *T* is a punctual regulated transformation defined as $T: \begin{cases} u = x^2 + y + z \\ v = x + y^2 + z \\ w = x + y + z^2 \end{cases}$.

We want to find the form of the equation $\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} - \frac{\partial^2 w}{\partial u \partial v} = 0$, with partial derivatives by utilizing the partial derivatives of function $z = z(x,y)$. By calculation we obtain:

$$F_{x^2} = 2;\ G_{x^2} = 0;\ H_{x^2} = 0;$$
$$F_{y^2} = 0;\ G_{y^2} = 2;\ H_{y^2} = 0;$$
$$F_{z^2} = 0;\ G_{z^2} = 0;\ H_{z^2} = 2;$$
$$F_x = 2x + z'_x;\ F_y = 1 + z'_y;$$
$$G_x = 1 + z'_x;\ G_y = 2y + z'_y;$$

$$\frac{\partial^2 w}{\partial u^2}=\frac{\begin{vmatrix}0 & 0 & (2x+z_x')(1+z_x')\\ 0 & 2 & (1+z_y')(2y+z_y')\\ 2z_{xy}'' & 0 & (1+z_x')(1+z_x')\end{vmatrix}}{\begin{vmatrix}2 & 0 & (2x+z_x')(1+z_x')\\ 0 & 2 & (1+z_y')(2y+z_y')\\ 0 & 0 & (1+z_y')(1+z_x')\end{vmatrix}}=\frac{-z_{xy}''(2x+z_x')(1+z_x')}{(1+z_y')(1+z_x')};$$

$$\frac{\partial^2 w}{\partial v^2}=\frac{\begin{vmatrix}2 & 0 & (2x+z_x')(1+z_x')\\ 0 & 0 & (1+z_y')(2y+z_y')\\ 0 & 2z_{xy}'' & (1+z_y')(1+z_x')\end{vmatrix}}{4(1+z_y')(1+z_x')}=\frac{-z_{xy}''(1+z_y')(2y+z_y')}{(1+z_y')(1+z_x')};$$

$$\frac{\partial^2 w}{\partial u\partial v}=\frac{\begin{vmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2z_{xy}''\end{vmatrix}}{4(1+z_y')(1+z_x')}=\frac{2z_{xy}''}{(1+z_y')(1+z_x')}.$$

Taking into account these equalities, the differential equation become:

$$(*)z_{xy}''\left((2x+z_x')(1+z_x')+(1+z_y')(2y+z_y')+2\right)=0$$

$$\Leftrightarrow z_{xy}''=0 \text{ or } (z_x')^2+(1+2x)z_x'+(z_y')^2+(1+2y)z_y'+2x+2y+2=0;$$

Conclusions: Finally, it can be observed that this punctual regular linear transformation reduce differential equation with partial derivatives of second order to differential equation by first order with partial derivatives.

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