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APPROXIMATE CONVEXITY IN MULTIOBJECTIVE PROGRAMMING

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Abstract: *We define a general class of approximate pseudo/quasi-convex functions. Using these functions, we establish necessary and sufficient optimality conditions for the quasi efficient solutions and higher order quasi efficient solutions of a multiobjective programming problem. Some duality properties of the problem and its mixed dual are also considered under generalized approximate convexity assumptions.*
Keywords: *multiobjective programming, approximate convexity, efficient solution, optimality, duality.*

1. INTRODUCTION

There have been several studies in the past to demonstrate the key role played by duality in economics and optimization theory. Many dual models have been proposed for the constrained vector optimization problems and corresponding duality results have been investigated.

It is worth to note that the notions of convexity and generalized convexity play a crucial role in establishing the primal-dual relationships. Moreover, advances in non-smooth analysis and non-smooth sub-differential calculus rules led various authors to search for the class of non-convex functions possessing properties that are this context. Ngai, Luc and Thera [6] defined a new class of approximate convex functions and showed that functions belonging to this class enjoy many of the desired properties.

The concept of several solution concepts in multiobjective programming has been widely studied in the literature [1, 2, 4, 5].

In this article, we use the notion of approximate convexity to obtain some duality results for non-smooth multiobjective optimizations problems.

In section 2, we present some preliminary facts concerning generalized approximate convex function in terms of Clarke generalized gradient. In section 3, we establish necessary and sufficient optimality conditions for the quasi efficient solutions and higher order quasi efficient solutions of a multiobjective programming problem. Finally, we present in section 4 some duality properties of the multiobjective problem and its mixed dual under generalized approximate convexity assumptions.

2. PRELIMINARIES

We denote by $\mathbf{R}_+^n = \left\{ x \in \mathbf{R}^n \mid x_i \geqq 0, \ \forall\, i \in I_n = \{1,...,n\} \right\}$ the non-negative orthant of the *n*-dimensional Euclidean space $\mathbf{R}^n$.

The null vector is denoted by **0** and $\mathbf{e} = (1,1,...,1)$. Both vectors have the dimension of the context they appear.

For any $x, y \in \mathbf{R}^n$, we use following notation:

$$x < y \iff \begin{cases} y - x \in \operatorname{int}\left(\mathbf{R}_+^n\right) \\ \left(i.e.\ x_i < y_i \ \ \forall\, i \in I_n\right) \end{cases}$$

$$x \leqq y \iff \begin{cases} y - x \in \mathbf{R}_+^n \\ \left(i.e.\ x_i \leqq y_i \ \ \forall\, i \in I_n\right) \end{cases}$$

$$x \leq y \iff \begin{cases} y - x \in \mathbf{R}_+^n \setminus \{\mathbf{0}\} \\ \left(i.e.\ x \leqq y \text{ and } \exists\, i \in I_n \text{ with } x_i < y_i\right) \end{cases}$$

$$x \nleq y \iff \begin{cases} y - x \notin \mathbf{R}_+^n \setminus \{\mathbf{0}\} \\ \left(i.e.\ x = y \text{ or } \exists\, i \in I_n \text{ with } x_i > y_i\right) \end{cases}$$

$$x^{\mathrm{T}} y = y^{\mathrm{T}} x = \sum_{i=1}^{n} x_i y_i \quad \text{(the standard inner product)}$$

For $x \in \mathbf{R}^n$ and $r \in \mathbf{R}_+$ we denote $\mathbf{B}(x, r) = \left\{ y \in \mathbf{R}^n \mid \|x - y\| \leqq r \right\}$.

For a given open set $X \subseteq \mathbf{R}^n$ we consider the multi-objective problem

$$\min_{x \in X} \left\{ f(x) \mid g(x) \leqq \mathbf{0} \right\} \qquad \text{(P)}$$

where $f = \left(f_1, \dots, f_p\right) : X \to \mathbf{R}^p,\ g = \left(g_1, \dots, g_q\right) : X \to \mathbf{R}^q.$

We denote the set of feasible solutions of problem (P) by

$$\mathscr{P} = \left\{ x \in X \subseteq \mathbf{R}^n \mid g(x) \leqq \mathbf{0} \right\}.$$

We use the following solution concepts for the problem (P):

Definition 2.1 A point $x_0 \in \mathscr{P}$ is an efficient solution for (P) if

$$\forall\, x \in \mathscr{P} \quad \Rightarrow \quad f(x) \nleq f(x_0).$$

Definition 2.2 A point $x_0 \in \mathscr{P}$ is a quasi-efficient solution for (P) if there exists $\alpha \in \operatorname{int}\left(\mathbf{R}_+^p\right)$ such that

$$\forall\, x \in \mathscr{P} \quad \Rightarrow \quad f(x) + \alpha \|x - x_0\| \nleq f(x_0).$$

Definition 2.3 A point $x_0 \in \mathscr{P}$ is a quasi-efficient solution of order *m* for (P) $(m > 1)$ if there exists $\beta \in \operatorname{int}\left(\mathbf{R}_+^p\right)$ such that

$$\forall\, x \in \mathscr{P} \quad \Rightarrow \quad f(x) + \beta \|x - x_0\|^m \nleq f(x_0).$$

Remark 2.1 Every efficient solution is a quasi-efficient solution, and a quasi-efficient solution of order *m* for (P), but the converses may not be true.

Definition 2.4 A point $x_0 \in \mathscr{P}$ is said to be a $(1, m)$-quasi-efficient solution for (P) $(m > 1)$ if there exist $\alpha, \beta \in \operatorname{int}\left(\mathbf{R}_+^p\right)$ such that

$$\forall\, x \in \mathscr{P} \quad \Rightarrow \quad f(x) + \alpha \|x - x_0\| + \beta \|x - x_0\|^m \nleq f(x_0).$$

Remark 2.2 Every quasi-efficient solution, and a quasi-efficient solution of order *m* for (P) is also a $(1, m)$-quasi-efficient solution for (P), but the converses may not be true.

The locally Lipschitz condition and Clarke generalized gradient are frequently used in analyzing non-smooth multi-objective optimization problems. For the sake of completeness we recall these definitions.

Definition 2.5 The function $\varphi : X \subseteq \mathbf{R}^n \to \mathbf{R}$ is locally Lipschitz at $x \in X$ if there $\exists\, L > 0$ and a neighborhood U_x of *x* such that

$$\left|\varphi(x_1) - \varphi(x_2)\right| \leqq L \|x_1 - x_2\|, \quad \forall x_1, x_2 \in U_x.$$

Definition 2.6 Let φ be locally Lipschitz at $x \in X$. The Clarke generalized directional derivative of φ at $\underline{x}$ in the direction $v \in \mathbf{R}^n$ is given by

$$\varphi^0(x, v) = \limsup_{\lambda \downarrow 0,\ y \to x} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

The locally Lipschitz condition assure the existence and finiteness of $\varphi^0(x, v)$. Moreover, as a function, $\varphi^0(x, \cdot)$ is sub-additive and positively homogeneous in the second argument. These properties together with the Hahn-Banach theorem give consistence to the following definition.

Definition 2.7 ([3]) The Clarke generalized gradient of φ at $x \in X$ is defined by

$$\partial\varphi(x) = \left\{ c \in \mathbf{R}^n \mid \varphi^0(x, v) \geqq c^{\mathrm{T}} v,\ \forall\, v \in \mathbf{R}^n \right\}.$$

It is worth to mention that the function $\varphi(x) = \|x - x_0\|$ is not differentiable at x_0, but its Clarke generalized gradient at x_0 is the closed unit ball $\mathbf{B}(\mathbf{0}, 1) \subseteq \mathbf{R}^n$. However, if we consider $\varphi(x) = \|x - x_0\|^m$ with $m > 1$, then the Clarke generalized gradient at x_0 is the null vector: $\partial\varphi(x_0) = \mathbf{0} \in \mathbf{R}^n$.

We enlarge the class of approximate convex functions introduced by Ngai, Luc and Thera [6].

Definition 2.8 A function $\varphi : X \to \mathbf{R}$ is $(\gamma; 1, m)$-approximate convex at $x_0 \in X$, where $\gamma > 0$ and $m > 1$, if $\forall \delta_1, \delta > 0,\ \exists\, r > 0$ (*r* depends on δ_1, δ and x_0) such that

$$\gamma\varphi\left(\lambda x+(1-\lambda)y\right)\leqq \lambda\varphi(x)+(\gamma-\lambda)\varphi(y)+$$
$$+\lambda(1-\lambda)\left(\delta_1\|x-y\|+\delta\|x-y\|^m\right), \qquad (2.1)$$
$$\forall\, x,y\in\mathbf{B}(x_0,r)\cap X,\ \forall\,\lambda\in\left(0,\min\{\gamma,1\}\right).$$

Remark 2.3 For $\gamma=1$ and $m=1$, the function is called approximate convex [6]. A lower semicontinuous approximate convex function at x_0 is locally Lipschitz at x_0 [6, Proposition 3.2]. This property also applies to lower semicontinuous $(\gamma;1,m)$-approximate convex functions.

Proposition 2.1 *Suppose that* $\varphi : X\to\mathbf{R}$ *is a proper lower semicontinuous function. If* φ *is* $(\gamma;1,m)$*-approximate convex at* $x_0\in X$ *, then* φ *is locally Lipschitz at* x_0*.*

Proof Since φ is $(\gamma;1,m)$-approximate convex at $x_0\in X$, there exist $r>0$ such that $\mathbf{B}(x_0,r)\subset X$, and for $\forall\, x,y\in\mathbf{B}(x_0,r)$ and $\forall\lambda\in\left(0,\min\{\gamma,1\}\right)$ we have

$$\varphi\left(\lambda x+(1-\lambda)y\right)\leqq \frac{\lambda}{\gamma}\varphi(x)+\left(1-\frac{\lambda}{\gamma}\right)\varphi(y)+$$
$$+\lambda(1-\lambda)\left(\frac{\delta_1}{\gamma}\|x-y\|+\frac{\delta}{\gamma}\|x-y\|^m\right).$$

We state that φ is locally bounded at x_0. To show this, let

$$U_n=\left\{x\in\mathbf{B}(x_0,r)\mid\varphi(x)\leqq n\right\},\quad n\in\mathbf{R}.$$

It follows that $\mathbf{B}(x_0,r)=\bigcup_{n\in\mathbf{N}}U_n$ and all U_n are closed. According to the Baire category theorem there is some index n_0 such that the interior of U_{n_0}, denoted by $\operatorname{int}\left(U_{n_0}\right)$, is nonempty. Let $z_0\in\operatorname{int}\left(U_{n_0}\right)$ and $\alpha>\max\left\{\frac{1}{\gamma},1\right\}$ such that

$$y_0 := z_0+\alpha\left(x_0-z_0\right)\in\operatorname{int}\left(U_{n_0}\right)$$

and select some nonnegative number $\rho<r$ such that $\forall\, x\in\mathbf{B}(x_0,\rho)$ one has $z := y_0+\alpha\left(x-y_0\right)\in\operatorname{int}\left(U_{n_0}\right)$. We have:

$$\varphi(x)=\varphi\left(\frac{1}{\alpha}z+\left(1-\frac{1}{\alpha}\right)y_0\right)\leqq$$
$$\leqq\frac{1}{\gamma\alpha}\varphi(z)+\left(1-\frac{1}{\gamma\alpha}\right)\varphi(y_0)+\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)\left(\frac{\delta_1}{\gamma}\|z-y_0\|+\frac{\delta}{\gamma}\|z-y_0\|^m\right)$$
$$\leqq\frac{n_0}{\gamma\alpha}+\left(1-\frac{1}{\gamma\alpha}\right)n_0+\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)\left(\frac{\delta_1}{\gamma}2r+\frac{\delta}{\gamma}(2r)^m\right)=V.$$

This means that φ is bounded from above on $\mathbf{B}(x_0,\rho)$ by the value *V*.

The function φ is also locally bounded from below. Indeed, for $\forall\, x\in\mathbf{B}(x_0,\rho)$ one has $2x_0-x\in\mathbf{B}(x_0,\rho)$ and we have

$$\varphi(x_0)\leqq\frac{1}{2\gamma}\varphi(x)+\left(1-\frac{1}{2\gamma}\right)\varphi(2x_0-x)+\frac{\delta_1}{4\gamma}\|x_0-x\|+\frac{\delta}{4\gamma}\|x_0-x\|^m.$$

Consequently, for $\forall\, x \in \mathbf{B}(x_0, \rho)$,

$$\varphi(x) \geqq 2\gamma\varphi(x_0) - (2\gamma - 1)V - \frac{\delta_1}{2}\|x_0 - x\| - \frac{\delta}{2}\|x_0 - x\|^m .$$

It follows now that there exists a margin *M* such that $|\varphi(x)| \leqq M$ for $\forall\, x \in \mathbf{B}(x_0, \rho)$.

For any $x, y \in \mathbf{B}\left(x_0, \frac{\rho}{2}\right)$, $x \neq y$, we denote $\eta = \|x - y\|$. We have $z := x + \frac{\rho}{2\eta}(x - y) \in \mathbf{B}(x_0, \rho)$. Hence,

$$\varphi(x) = \varphi\left(\frac{2\eta}{\rho + 2\eta} z + \frac{\rho}{\rho + 2\eta} y\right) \leqq \frac{2\eta}{\gamma(\rho + 2\eta)}\varphi(z) + \left(1 - \frac{2\eta}{\gamma(\rho + 2\eta)}\right)\varphi(y)$$
$$+ \frac{2\eta}{\rho + 2\eta}\left(1 - \frac{2\eta}{\rho + 2\eta}\right)\left(\frac{\delta_1}{\gamma}\|z - y\| + \frac{\delta}{\gamma}\|z - y\|^m\right).$$

It follows

$$\varphi(x) - \varphi(y) \leqq \frac{2\eta}{\gamma(\rho + 2\eta)}\left(\varphi(z) - \varphi(y)\right) +$$
$$+ \frac{2\eta}{\rho + 2\eta}\frac{\rho}{\rho + 2\eta}\left(\frac{\delta_1}{\gamma}\|z - y\| + \frac{\delta}{\gamma}\|z - y\|^m\right)$$

Since obviously $\frac{2\eta}{\gamma(\rho + 2\eta)} \leqq \frac{2\eta}{\gamma\rho}$, $\frac{\rho}{\rho + 2\eta} < 1$, $\eta = \|x - y\| \leqq \rho$ and $\|z - y\| \leqq \frac{3\rho}{2}$, we get

$$\varphi(x) - \varphi(y) \leqq \frac{2\eta}{\gamma\rho}\left(\varphi(z) - \varphi(y)\right) + \frac{2\eta}{\gamma\rho}\left(\delta_1\|z - y\| + \delta\|z - y\|^m\right) \leqq$$
$$\leqq \frac{2\eta}{\gamma\rho}2M + \frac{2\eta}{\gamma\rho}\left(\delta_1\frac{3\rho}{2} + \delta\left(\frac{3\rho}{2}\right)^m\right) = L\|x - y\|,$$

where $L = \frac{4M}{\gamma\rho} + \frac{3}{\gamma}\left(\delta_1 + \delta\left(\frac{3\rho}{2}\right)^{m-1}\right)$. Interchanging *x* and *y*, we obtain finally

$$|\varphi(x) - \varphi(y)| \leqq L\|x - y\|. \qquad \square$$

We present a characterization of $(\gamma;1,m)$-approximate convex functions in terms of the Clarke generalized gradient.

Theorem 2.2 *If* $\varphi : X \to \mathbf{R}$ *is a lower semicontinuous* $(\gamma;1,m)$*-approximate convex function at* $x_0 \in X$*, then* $\forall\,\delta_1, \delta > 0$, $\exists\, r > 0$ *such that*

$$\varphi(y) - \varphi(x_0) \geqq \gamma c^{\mathrm{T}}(y - x_0) - \delta_1\|y - x_0\| - \delta\|y - x_0\|^m,$$
$$\forall\, y \in \mathbf{B}(x_0, r) \cap X, \quad \forall\, c \in \partial\varphi(x_0).$$

Proof The assumption implies that φ is locally Lipschitz at x_0 and (2.1) holds for $\forall\,\delta_1, \delta > 0$ and some $r > 0$. Let $y \in \mathbf{B}(x_0, r) \cap X$ and $h > \mathbf{0}$ sufficiently small so that $x_0 + h$ and $y + h \in \mathbf{B}(x_0, r)$.

The Clarke generalized directional derivative of φ at x_0 along $(y - x_0)$ is

$$\varphi^0(x, y-x_0) = \limsup_{\lambda \downarrow 0,\, h\to 0} \frac{\varphi((x_0+h)+\lambda(y-x_0)) - \varphi(x_0+h)}{\lambda} =$$

$$= \limsup_{\lambda \downarrow 0,\, h\to 0} \frac{\varphi(\lambda(y+h)+(1-\lambda)(x_0+h)) - \varphi(x_0+h)}{\lambda} \leqq$$

$$\leqq \limsup_{\lambda \downarrow 0,\, h\to 0} \left[\frac{1}{\gamma}\varphi(y+h) - \frac{1}{\gamma}\varphi(x_0+h) + \frac{1}{\gamma}\delta_1(1-\lambda)\|y-x_0\| + \frac{1}{\gamma}\delta(1-\lambda)\|y-x_0\|^m\right] \leqq$$

$$\leqq \frac{1}{\gamma}\limsup_{\lambda \downarrow 0,\, h\to 0}\left[\varphi(y)+L\|h\| - \varphi(x_0) + L\|h\| + \delta_1(1-\lambda)\|y-x_0\| + \delta(1-\lambda)\|y-x_0\|^m\right] =$$

$$= \frac{1}{\gamma}\left[\varphi(y)-\varphi(x_0)+\delta_1\|y-x_0\|+\delta\|y-x_0\|^m\right].$$

Since $\gamma > 0$, for $\forall\, c \in \partial\varphi(x_0)$ we have:

$$\gamma c^{\mathrm{T}}(y-x_0) \leqq \gamma\varphi^0(x, y-x_0) \leqq \varphi(y)-\varphi(x_0)+\delta_1\|y-x_0\|+\delta\|y-x_0\|^m. \square$$

Definition 2.9 A function $\varphi : X \to \mathbf{R}$ is $(\gamma;1,m)$-approximate quasi-convex at $x_0 \in X$, where $\gamma > 0$ and $m > 1$, if $\forall \delta_1, \delta > 0$, $\exists r > 0$ (*r* depends on δ_1, δ and x_0) such that

$$\begin{aligned} &\forall\, y \in \mathbf{B}(x_0, r) \cap X, \ \text{ with } \ \varphi(y) \leqq \varphi(x_0), \quad \Rightarrow \\ &\gamma c^{\mathrm{T}}(y-x_0) - \delta_1\|y-x_0\| - \delta\|y-x_0\|^m \leqq 0, \ \ \forall\, c \in \partial\varphi(x_0). \end{aligned} \quad (2.2)$$

Definition 2.10 A function $\varphi : X \to \mathbf{R}$ is $(\gamma;1,m)$-approximate pseudo-convex at $x_0 \in X$, where $\gamma > 0$ and $m > 1$, if $\forall \delta_1, \delta > 0$, $\exists r > 0$ (*r* depends on δ_1, δ and x_0) such that

$$\left.\begin{aligned} &\forall\, y \in \mathbf{B}(x_0, r) \cap X, \ \exists\, c \in \partial\varphi(x_0) \ \text{ with} \\ &\gamma c^{\mathrm{T}}(y-x_0) + \delta_1\|y-x_0\| + \delta\|y-x_0\|^m \geqq 0, \end{aligned}\right\} \Rightarrow \quad (2.3)$$

$$\Rightarrow \ \varphi(y) + \delta_1\|y-x_0\| + \delta\|y-x_0\|^m \geqq \varphi(x_0).$$

Remark 2.4 A $(\gamma;1,m)$-approximate convex function at x_0 is both $(\gamma;1,m)$-approximate quasi-convex and $(\gamma;1,m)$-approximate pseudo-convex at x_0, but the converses do not hold in general.

3. OPTIMALITY CONDITIONS

A necessary optimality conditions for quasi-efficient solutions is given by the following theorem, which is an extension of [5, Theorem 2].

Theorem 3.1 *Let $x_0 \in \mathcal{P}$ be an $(1,m)$-quasi-efficient solution for problem (P). If the component functions of f and g are locally Lipschitz at x_0, then for any $\gamma \in \operatorname{int}(\mathbf{R}^p_+)$, $\tau \in \operatorname{int}(\mathbf{R}^q_+)$, there exist the vectors $\alpha \in \operatorname{int}(\mathbf{R}^p_+)$, $\lambda \in \mathbf{R}^p_+$ and $\mu \in \mathbf{R}^q_+$ such that*

$$\mathbf{0} \in \sum_{i=1}^{p}\lambda_i\gamma_i\partial f_i(x_0) + \sum_{j=1}^{q}\mu_j\tau_j\partial g_j(x_0) + \sum_{i=1}^{p}\lambda_i\gamma_i\alpha_i\mathbf{B}(\mathbf{0},1) \quad (3.1)$$

$$\mu_j\tau_j g_j(x_0) = 0, \quad j = \overline{1,q}. \quad (3.2)$$

Proof Since $x_0 \in \mathcal{P}$ is an $(1,m)$-quasi-efficient solution for (P), it follows that there exist $\alpha, \beta \in \operatorname{int}(\mathbf{R}^p_+)$ such that for $\forall\, x \in \mathcal{P}$ the following system is incompatible:

$$\begin{cases} f(x) \leq f(x_0) - \alpha \|x - x_0\| - \beta \|x - x_0\|^m , \\ g(x) \leqq \mathbf{0} . \end{cases}$$

It follows that, for any $\gamma \in \operatorname{int}(\mathbf{R}_+^p)$, $\tau \in \operatorname{int}(\mathbf{R}_+^q)$, the system

$$\begin{cases} \gamma_i f_i(x) \leq \gamma_i f_i(x_0) - \gamma_i \alpha_i \|x - x_0\| - \gamma_i \beta_i \|x - x_0\|^m , \quad i = \overline{1, p} \ , \\ \tau_j g_j(x_0) ? 0, \quad j = \overline{1, q} \ , \end{cases}$$

is also incompatible for $\forall \ x \in \mathscr{P}$. Consequently, x_0 is an efficient solution of the following multi-objective problem:

$$\min \begin{Bmatrix} \gamma_1 f_1(x) + \gamma_1 \alpha_1 \|x - x_0\| + \gamma_1 \beta_1 \|x - x_0\|^m , \ ... \\ ... , \ \gamma_p f_p(x) + \gamma_p \alpha_p \|x - x_0\| + \gamma_p \beta_p \|x - x_0\|^m \end{Bmatrix}$$

$$\text{subject to} \quad \tau_j g_j(x_0) \leqq 0, \quad j = \overline{1, q}$$

Applying now Fritz-John necessary optimality conditions to this problem, we get the existence of $\lambda \in \mathbf{R}_+^p$ and $\mu \in \mathbf{R}_+^q$ such that

$$\mathbf{0} \in \sum_{i=1}^{p} \lambda_i \partial \left(\gamma_i f_i + \gamma_i \alpha_i \|x - x_0\| + \gamma_i \beta_i \|x - x_0\|^m \right)(x_0) + \sum_{j=1}^{q} \mu_j \tau_j \partial g_j(x_0)$$

$$\mu_j \tau_j g_j(x_0) = 0, \quad j = \overline{1, q} \ ,$$

$$(\lambda, \mu) \neq \mathbf{0} \ .$$

But these relations are equivalent to (3.1) and (3.2) because

$$\partial \left(\gamma_i f_i + \gamma_i \alpha_i \|x - x_0\| + \gamma_i \beta_i \|x - x_0\|^m \right)(x_0) = \gamma_i \partial f_i(x_0) + \gamma_i \alpha_i \mathbf{B}(\mathbf{0}, 1) \ .$$

Remark 3.1 The necessary optimality condition developed above are Fritz-John type. Under appropriate constraint qualifications or regularity conditions on the functions we can easily derive the KKT type necessary optimality conditions. In that case we can take $\lambda^{\mathrm{T}} \mathbf{e} = 1$. One such constraint qualification is Mangasarian Fromovitz constraint qualification which states that

$$\mathbf{0} \in \sum_{j \in J(x_0)} \mu_j \partial g_j(x_0) \ \Rightarrow \ \mu_j = 0, \ \forall j \in J(x_0) = \left\{ j \mid g_j(x_0) = 0 \right\} .$$

Another weakened form of constraint qualification called basic regularity condition is given as follows

$$\mathbf{0} \in \sum_{i=1, \ i \neq s}^{p} \lambda_i \partial f_i(x_0) + \sum_{j \in J(x_0)} \mu_j \partial g_j(x_0) + \sum_{i=1, \ i \neq s}^{p} \lambda_i \alpha_i \mathbf{B}(\mathbf{0}, 1) \quad \text{for some } s$$

$$\Rightarrow \lambda_i = 0, \ \forall \ i \in \{1, ..., p\} \setminus \{s\}, \ \mu_j = 0, \ \forall \ j \in J(x_0) \ .$$

The next theorem states a sufficient optimality condition.

Theorem 3.2 *We assume that the conditions (3.1) and (3.2) are satisfied at* $x_0 \in \mathscr{P}$ *and* $\lambda \in \operatorname{int}(\mathbf{R}_+^p)$, $\lambda^{\mathrm{T}} \mathbf{e} = 1$. *If the component functions of f and g are* $(\gamma_i; 1, m)$*-approximate, respectively* $(\tau_j; 1, m)$*-approximate convex at* x_0*, then* x_0 *is a local* $(1, m)$*-quasi-efficient solution for (P).*

Proof Since the conditions (3.1) and (3.2) are satisfied at x_0, it follows that for some $c_i \in \partial f_i(x_0)$, $d_j \in \partial g_j(x_0)$, and $b \in \mathbf{B}(\mathbf{0}, 1)$ we have

$$\mathbf{0} = \sum_{i=1}^{p} \lambda_i \gamma_i c_i + \sum_{j=1}^{q} \mu_j \tau_j d_j + \sum_{i=1}^{p} \lambda_i \gamma_i \alpha_i b \qquad (3.3)$$

$$\mu_j \tau_j g_j(x_0) = 0, \quad j = \overline{1, q} \ . \qquad (3.4)$$

According to Theorem 2.2, $\forall \delta_1, \delta > 0$, $\exists \ r > 0$ such that $\forall \ x \in \mathbf{B}(x_0, r)$ we have

$$f_i(x) - f_i(x_0) \geqq \gamma_i c_i^{\mathrm{T}}(x - x_0) - \delta_1 \|x - x_0\| - \delta \|x - x_0\|^m, \quad \forall\, c_i \in \partial f_i(x_0),$$

$$g_j(x) - g_j(x_0) \geqq \tau_j d_j^{\mathrm{T}}(x - x_0) - \delta_1 \|x - x_0\| - \delta \|x - x_0\|^m, \quad \forall\, \beta_j \in \partial g_j(x_0).$$

Using (3.3) and (3.4), since $\lambda > \mathbf{0}$, and $\mu \geqq \mathbf{0}$, the two above inequalities yield

$$\lambda^{\mathrm{T}}\left(f(x) - f(x_0)\right) + \mu^{\mathrm{T}} g(x) \geqq$$

$$\geqq \left(\sum_{i=1}^{p} \lambda_i \gamma_i c_i + \sum_{j=1}^{q} \mu_j \tau_j d_j\right)^{\mathrm{T}} (x - x_0) - \left(1 + \mu^{\mathrm{T}}\mathbf{e}\right)\left[\delta_1 \|x - x_0\| + \delta \|x - x_0\|^m\right]$$

$$= -\left(\sum_{i=1}^{p} \lambda_i \gamma_i \alpha_i b\right)^{\mathrm{T}} (x - x_0) - \left(1 + \mu^{\mathrm{T}}\mathbf{e}\right)\left[\delta_1 \|x - x_0\| + \delta \|x - x_0\|^m\right]$$

$$\geqq -\sum_{i=1}^{p} \lambda_i \gamma_i \alpha_i \|x - x_0\| - \left(1 + \mu^{\mathrm{T}}\mathbf{e}\right)\left[\delta_1 \|x - x_0\| + \delta \|x - x_0\|^m\right]$$

$$= -\eta_{\delta_1} \|x - x_0\| - \eta_\delta \|x - x_0\|^m,$$

where $\eta_{\delta_1} = \sum_{i=1}^{p} \lambda_i \gamma_i \alpha_i + \left(1 + \mu^{\mathrm{T}}\mathbf{e}\right)\delta_1 > 0$ and $\eta_\delta = \left(1 + \mu^{\mathrm{T}}\mathbf{e}\right)\delta > 0$.

Now, for $\forall x \in \mathbf{B}(x_0, r) \cap \mathscr{P}$ we have

$$\lambda^{\mathrm{T}}\left(f(x) - f(x_0)\right) + \eta_{\delta_1} \|x - x_0\| + \eta_\delta \|x - x_0\|^m \geqq -\mu^{\mathrm{T}} g(x) \geqq 0,$$

and therefore we get

$$\lambda^{\mathrm{T}}\left(f(x) - f(x_0) + \eta_{\delta_1} \|x - x_0\| + \eta_\delta \mathbf{e} \|x - x_0\|^m\right) \geqq 0.$$

But this relation imply that $\forall \delta_1, \delta > 0$, $\exists\, r > 0$ and $\eta_{\delta_1} > 0,\ \eta_\delta > 0$ such that $\forall x \in \mathbf{B}(x_0, r) \cap \mathscr{P}$ we cannot have

$$f(x) \leq f(x_0) - \eta_{\delta_1} \|x - x_0\| - \eta_\delta \mathbf{e} \|x - x_0\|^m,$$

i.e., x_0 is a local $(1, m)$-quasi-efficient solution for (P).

4. DUALITY

We study now the duality relationship between the problem (P) and its mixed dual under generalized approximate convexity assumptions.

The constraints index set $\{1, \ldots, q\} = J_0 \cup J_1$ is partitioned in the two disjoint subsets J_0 and J_1 and we denote $\mathbf{e} = (1, \ldots, 1)^{\mathrm{T}} \in \mathbf{R}^p$.

The mixed dual is the following problem:

$$\max \left\{ f(u) + \mu_{J_0}^{\mathrm{T}} g_{J_0}(u)\mathbf{e} \right\} \qquad \text{(D)}$$

subject to

$$\left.\begin{array}{c} \mathbf{0} \in \partial \lambda^{\mathrm{T}} f(u) + \partial \mu^{\mathrm{T}} g(u) + \lambda^{\mathrm{T}} \alpha \mathbf{B}(\mathbf{0}, 1) \\ \mu_j g_j(u) \geqq 0, \quad j \in J_1 \\ \lambda \geqq \mathbf{0},\ \lambda^{\mathrm{T}}\mathbf{e} = 1,\ \mu \geqq \mathbf{0},\ \alpha > \mathbf{0}. \end{array}\right\} \qquad (4.1)$$

We denote the set of feasible solutions of the dual problem (D) by

$$\mathscr{D} = \left\{ (u, \lambda, \mu, \alpha) \text{ that satisfy (4.1)} \right\}.$$

Theorem 4.1 (week duality) *Let* $(u, \lambda, \mu, \alpha) \in \mathscr{D}$ *and suppose* $\mu_{J_1}^{\mathrm{T}} g_{J_1}(\cdot)$ *is* $(\tau; 1, m)$*-approximate quasi-convex and* $\left(\lambda^{\mathrm{T}} f + \mu_{J_0}^{\mathrm{T}} g_{J_0}\right)(\cdot)$ *is* $(\gamma; 1, m)$*-approximate pseudo-convex at u. Then* $\forall\, \delta_1 > 2\gamma\lambda^{\mathrm{T}}\alpha$ *and* $\forall\, \delta > 0$*, there exists* $\bar{r} > 0$ *such that the following does not hold*

$$f_i(x) < f_i(u) + \mu_{J_0}^{\mathrm{T}} g_{J_0}(u) - \delta_1 \|x-u\| - \delta \|x-u\|^m, \quad \forall\, i \in \{1,\dots,p\}$$

where $x = u + tv,\ v \in \mathbf{R}^n,\ 0 < t < \bar{r}$ *, and* $x \in \mathscr{P}$.

Proof Since $(u,\lambda,\mu,\alpha) \in \mathscr{D}$, for some $c_i \in \partial\lambda_i f_i(u)$, $d_j \in \partial\mu_j g_j(u)$, and $b \in \mathbf{B}(\mathbf{0},1)$ we have

$$\mathbf{0} = \sum_{i=1}^{p} c_i + \sum_{j=1}^{m} d_j + \sum_{i=1}^{p} \lambda_i \alpha_i b \,. \qquad (4.2)$$

Let $x \in \mathscr{P}$. Since $\mu \geqq \mathbf{0}$, in particular we have

$$\sum_{j\in J_1} \mu_j^{\mathrm{T}} g_j(x) \leqq \sum_{j\in J_1} \mu_j^{\mathrm{T}} g_j(u) \qquad (4.3)$$

Using the $(\tau;1,m)$-approximate quasi-convexity of $\mu_{J_1}^{\mathrm{T}} g_{J_1}(\cdot)$ at *u*, for $\forall\ \delta_1', \delta' > 0\ \exists\ r > 0$ such that whenever $x \in \mathbf{B}(u,r) \cap \mathscr{P}$ and (4.3) holds, we have

$$\sum_{j\in J_1} \tau d_j^{\mathrm{T}} (x-u) - \delta_1' \|x-u\| - \delta' \|x-u\|^m \leqq 0 \,.$$

Without loss of generality we can assume $\|v\| = 1$. Choosing $x = u + tv \in \mathscr{P},\ 0 < t < r$, the above arguments along with (4.2) yields

$$\left(\sum_{i=1}^{p} c_i + \sum_{j\in J_0} d_j + \sum_{i=1}^{p} \lambda_i \alpha_i b \right)^{\mathrm{T}} (x-u) + \frac{\delta_1'}{\tau} \|x-u\| + \frac{\delta'}{\tau} \|x-u\|^m \geqq 0 \,.$$

If we set $\dfrac{\delta_1}{\gamma} = \sum_{i=1}^{p} \lambda_i \alpha_i + \dfrac{\delta_1'}{\tau} > 0$ and $\dfrac{\delta}{\gamma} = \dfrac{\delta'}{\tau}$ we obtain

$$\gamma \left(\sum_{i=1}^{p} c_i + \sum_{j\in J_0} d_j \right)^{\mathrm{T}} (x-u) + \delta_1 \|x-u\| + \delta \|x-u\|^m \geqq 0 \,. \ (4.4)$$

Using now the $(\gamma;1,m)$-approximate pseudo-convexity of $\left(\lambda^{\mathrm{T}} f + \mu_{J_0}^{\mathrm{T}} g_{J_0}\right)(\cdot)$ at *u*, there exists $r' > 0$ such that whenever $x \in \mathbf{B}(u,r') \cap \mathscr{P}$ and (4.4) holds, we have

$$\lambda^{\mathrm{T}} f(x) + \mu_{J_0}^{\mathrm{T}} g_{J_0}(x) + \delta_1 \|x-u\| + \delta \|x-u\|^m \geqq \lambda^{\mathrm{T}} f(u) + \mu_{J_0}^{\mathrm{T}} g_{J_0}(u) \,.$$

If we take $\bar{r} = \min\{r, r'\}$, for $x = u + tv \in \mathscr{P},\ 0 < t < \bar{r}$, we obtain

$$\lambda^{\mathrm{T}} \left(f(x) - f(u) - \mu_{J_0}^{\mathrm{T}} g_{J_0}(u) \mathbf{e} + \left(\delta_1 \|x-u\| + \delta \|x-u\|^m \right) \mathbf{e} \right) \geqq 0 \,,$$

implying that

$$f_i(x) < f_i(u) + \mu_{J_0}^{\mathrm{T}} g_{J_0}(u) - \delta_1 \|x-u\| - \delta \|x-u\|^m, \quad \forall\, i \in \{1,\dots,p\}$$

is not possible.

Definition 4.1 $(u_0, \lambda_0, \mu_0, \alpha_0) \in \mathscr{D}$ is said to be a local weak $(1,m)$-quasi-efficient solution of (D) if there exist $\eta^1, \eta \in \operatorname{int}\left(\mathbf{R}_+^p\right)$ and a neighborhood U_0 of $(u_0, \lambda_0, \mu_0, \alpha_0)$ such that for any $(u,\lambda,\mu,\alpha) \in \mathscr{D} \cap U_0$ the following relation cannot hold

$$f_i(u_0) + \mu_{0J_0}^{\mathrm{T}} g_{J_0}(u_0) + \eta^1 \|u-u_0\| + \eta \|u-u_0\|^m < f_i(u) + \mu_{J_0}^{\mathrm{T}} g_{J_0}(u), \quad \forall\, i \in \{1,\dots,p\}.$$

Theorem 4.2 (strong duality) *Suppose* $x_0 \in \mathscr{P}$ *is a* $(1,m)$*-quasi-efficient solution of (P) and an approximate constraint qualification (like Mangasarian Fromovitz constaint qualification) or regularity condition (like basic regularity condition) is satisfied at* x_0*. Then there exist* $\alpha_0 \in \operatorname{int}\left(\mathbf{R}_+^p\right)$*,* $\lambda_0 \in \mathbf{R}_+^p$*,* $\mu_0 \in \mathbf{R}_+^m$ *such that* $(x_0, \lambda_0, \mu_0, \alpha_0) \in \mathscr{D}$*. Further, if the conditions of weak duality hold with* $\delta_1 > 2\gamma \sum_{i=1}^{p} \lambda_i \alpha_i$*, and* $\delta > 0$*, then* $(x_0, \lambda_0, \mu_0, \alpha_0)$ *is a local weak quasi-efficient solution of (D) and the objective values of (P) and (D) are equal.*

Proof According to Theorem 3.1 and Remark 3.1, there exist $\alpha \in \operatorname{int}\left(\mathbf{R}_{+}^{p}\right)$, $\lambda_0 \in \mathbf{R}_{+}^{p}$, $\lambda_0^{\mathrm{T}}\mathbf{e} = 1$, $\mu_0 \in \mathbf{R}_{+}^{m}$ such that $\left(x_0, \lambda_0, \mu_0, \alpha_0\right) \in \mathfrak{D}$. Moreover the objective value of (P) and (D) are equal to $f\left(x_0\right)$. Invoking the weak duality between (P) and (D), for every $\delta_1 > 2\gamma \sum_{i=1}^{p} \lambda_i \alpha_i$, and $\delta > 0$, there exist $\overline{r} > 0$ such that for any $u \in \mathbf{B}\left(x_0, \overline{r}\right)$, $x_0 = u + tv$, $0 < t < \overline{r}$, $v \in \mathbf{R}^n$, $\|v\| = 1$, the inequalities

$$f_i\left(x_0\right) = f_i\left(u + tv\right) < f_i\left(u\right) + \mu_{0J_0}^{\mathrm{T}} g_{J_0}\left(u\right) - \delta_1 \|x - u\| - \delta \|x - u\|^m, \quad \forall\, i \in \{1, ..., p\}$$

do not hold, implying that for any $u \in \mathbf{B}\left(x_0, \overline{r}\right)$ the inequalities

$$f_i\left(x_0\right) + \mu_{0J_0}^{\mathrm{T}} g_{J_0}\left(x_0\right) + \delta_1 \|u - x_0\| + \delta \|x - u\|^m < f_i\left(u\right) + \mu_{0J_0}^{\mathrm{T}} g_{J_0}\left(u\right), \quad \forall\, i \in \{1, ..., p\}$$

do not hold. Consequently $\left(x_0, \lambda_0, \mu_0, \alpha_0\right)$ is a local weak quasi-efficient solution of (D).

Remark 4.1 In Theorem 4.2, and thereby in Theorem 4.1, if $\lambda_0 > \mathbf{0}$ then we can show that $\left(x_0, \lambda_0, \mu_0, \alpha_0\right)$ is a local quasi-efficient solution of (D).

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