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THE INFLUENCE OF THE INITIAL CONDITIONS ON THE EVOLUTION OF PRESSURE WAVES

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Abstract*: In this paper it will be analyzed the influence of initial conditions on the evolution of the pressure wave using the linear and homogeneous wave equation (hyperbolic type) and MathCAD software.*

1. D'Alembert solution

Linear and homogeneous equation of second order with constant coefficients,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{a^2}\cdot\frac{\partial^2 u}{\partial t^2}, \quad (1)$$

hyperbolic type, called wave equation into space [1, 2], can be written as:

$\Box_3 u=0$, where the operator $\Box_3 u=\frac{1}{a^2}\cdot\frac{\partial^3 u}{\partial t^2}-\Delta_3 u$

The functions: $u_1(x,y,z,t)=\frac{1}{r}\cdot\varphi_1(t+\frac{r}{a})$, $u_2(x,y,z,t)=\frac{1}{r}\cdot\varphi_2(t-\frac{r}{a})$, (2)

where, $r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$, are the solutions of the equation (1), whatever functions φ_1 and φ_2, having the first and second order derivatives.

Solutions of the form (2) of equation (1) is called spherical wave and the solution as:

$$u(x,y,z,t)=\frac{1}{r}\cdot\varphi(r+a\cdot t)+\frac{1}{r}\cdot\psi(r-a\cdot t), \quad (3)$$

is named *D'Alembert solution*.

1.1 One-dimensional propagation

The wave equation [1, 2, 4] has the form: $\frac{1}{a^2}\cdot\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0$, (4)

to which are attached the initial conditions: $u(x,0)=\varphi(x)$, $u_t'(x,0)=\psi(x)$ (5)

where $\varphi(x)$ has to be twice differentiable and $\psi(x)$ once.

The solution of the Cauchy problem is:

$$u(x,t)=\frac{1}{2}\cdot\left[\varphi(x-a\cdot t)+\varphi(x+a\cdot t)\right]+\frac{1}{2\cdot a}\cdot\int_{x-a\cdot t}^{x+a\cdot t}\psi(\tau)\cdot d\tau. \quad (6)$$

1.2 Case study

We will analyze one dimensional propagation of the pressure which is given by the equation:

$$\frac{1}{a^2}\cdot\frac{\partial^2 p}{\partial t^2}-\frac{\partial^2 p}{\partial x^2}=0, \quad (CS.\ 1)$$

with boundary conditions: $p(x,0)=\varphi(x)$, $\frac{\partial p}{\partial t}(x,0)=\psi(x)$ (CS. 2)

where: $p0=10^8\left[\frac{N}{m^2}\right]$, $a=2$, $x0=0.5\ [m]$, and

$$\phi(x) := \begin{cases} 0 & \text{if } -10\cdot x0\le x\le -2\cdot x0 \\ \frac{p0}{x0}\cdot(x+2\cdot x0) & \text{if } -2\cdot x0\le x\le -x0 \\ p0 & \text{if } -x0\le x\le x0 \\ \frac{p0}{x0}\cdot(2\cdot x0-x) & \text{if } x0\le x\le 2\cdot x0 \\ 0 & \text{if } 2x0\le x\le 10\cdot x0 \end{cases} \quad (CS.3)$$

$$\psi(x):=0 \quad (CS.4)$$

The solution of the equation (CS.1), with conditions (SC.2) and (SC.3) is:

$$p(x,t)=\frac{1}{2}\cdot\left[\varphi(x-a\cdot t)+\varphi(x+a\cdot t)\right]+\frac{1}{2\cdot a}\cdot\int_{x-a\cdot t}^{x+a\cdot t}\psi(\tau)\cdot d\tau \quad (CS.5)$$

If $x := -5\cdot x0, -5\cdot x0+x0 \,..\, 5\cdot x0$, result:

$x =$	$p(x,0) =$	$p(x,0.1) =$	$p(x,0.2) =$	$p(x,0.3) =$	$p(x,0.4) =$	$p(x,0.5)$
-2.5	0	0	0	0	0	0
-2	0	0	0	0	0	0
-1.5	0	0	0	$1\cdot10^{7}$	$3\cdot10^{7}$	$5\cdot10^{7}$
-1	0	$2\cdot10^{7}$	$4\cdot10^{7}$	$5\cdot10^{7}$	$5\cdot10^{7}$	$5\cdot10^{7}$
-0.5	$1\cdot10^{8}$	$8\cdot10^{7}$	$6\cdot10^{7}$	$5\cdot10^{7}$	$5\cdot10^{7}$	$5\cdot10^{7}$
0	$1\cdot10^{8}$	$1\cdot10^{8}$	$1\cdot10^{8}$	$8\cdot10^{7}$	$4\cdot10^{7}$	0
0.5	$1\cdot10^{8}$	$8\cdot10^{7}$	$6\cdot10^{7}$	$5\cdot10^{7}$	$5\cdot10^{7}$	$5\cdot10^{7}$
1	0	$2\cdot10^{7}$	$4\cdot10^{7}$	$5\cdot10^{7}$	$5\cdot10^{7}$	$5\cdot10^{7}$
1.5	0	0	0	$1\cdot10^{7}$	$3\cdot10^{7}$	$5\cdot10^{7}$
2	0	0	0	0	0	0
2.5	0	0	0	0	0	0

Using the (CS.5) solution we obtain graphical representation as is shown in figure 1.

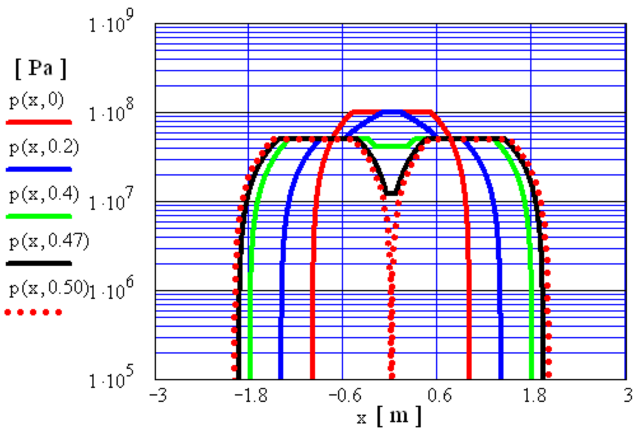

Figure 1 One-dimensional propagation of the pressure wave

To highlight the influence of initial condition $\psi(x)$ we consider the options:

$$\psi1(x) = 0$$

$$\psi2(x) = 10^3$$

$$\psi3(x) = 10^6$$

It will be obtained the following numerical values:

$x =$	$p1(x,0.2) =$	$p2(x,0.2) =$	$p3(x,0.2) =$
-1.5	0	200	$2\cdot10^{5}$
-1.25	$1.5\cdot10^{7}$	$1.5\cdot10^{7}$	$1.52\cdot10^{7}$
-1	$4\cdot10^{7}$	$4\cdot10^{7}$	$4.02\cdot10^{7}$
-0.75	$5\cdot10^{7}$	$5\cdot10^{7}$	$5.02\cdot10^{7}$
-0.5	$6\cdot10^{7}$	$6\cdot10^{7}$	$6.02\cdot10^{7}$
-0.25	$8.5\cdot10^{7}$	$8.5\cdot10^{7}$	$8.52\cdot10^{7}$
0	$1\cdot10^{8}$	$1\cdot10^{8}$	$1.002\cdot10^{8}$
0.25	$8.5\cdot10^{7}$	$8.5\cdot10^{7}$	$8.52\cdot10^{7}$
0.5	$6\cdot10^{7}$	$6\cdot10^{7}$	$6.02\cdot10^{7}$
0.75	$5\cdot10^{7}$	$5\cdot10^{7}$	$5.02\cdot10^{7}$
1	$4\cdot10^{7}$	$4\cdot10^{7}$	$4.02\cdot10^{7}$
1.25	$1.5\cdot10^{7}$	$1.5\cdot10^{7}$	$1.52\cdot10^{7}$
1.5	0	200	$2\cdot10^{5}$

The wave equation in space (1) with initial conditions of the Cauchy [3, 4]:

$$u(x,y,z,0) = f(x,y,z),\quad u_t'(x,y,z,0) = g(x,y,z), \tag{7}$$

has the solution:

$$u(x,y,z,t) = u_1(x,y,z,t) + u_2(x,y,z,t), \tag{8}$$

$$u_1(x,y,z,t) = \frac{1}{4\cdot\pi}\cdot\int_0^{2\pi}\int_0^{\pi} g(X,Y,Z)\cdot\sin(\theta)\cdot d\theta\cdot d\varphi, \tag{9}$$

$$u_2(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\cdot\pi}\cdot\int_0^{2\pi}\int_0^{\pi} f(X,Y,Z)\cdot\sin(\theta)\cdot d\theta\cdot d\varphi\right], \tag{10}$$

where: $X = x + a\cdot t\cdot\cos(\varphi)\cdot\sin(\theta)$, $Y = y + a\cdot t\cdot\sin(\varphi)\cdot\sin(\theta)$, $Z = z + a\cdot t\cdot\cos(\theta)$.

1.2 Case of spherical symmetry

In the case of spherical symmetry, the solution of the equation (1) with initial conditions:

$$p(r,0)=f(r),\ p_r^{'}(R,t)=0,\ p_t^{'}(r,o)=F(r), \qquad (11)$$

has the form [2, 3, 4]:

$$p(r,t)=\frac{3}{R}\cdot\int_0^R\left[f(\rho)+t\cdot F(\rho)\right]\cdot d\rho+$$
$$+\sum_{n=1}^{\infty}\left[a_n\cdot\cos\left(\frac{a\cdot\mu_n\cdot t}{R}\right)+b_n\cdot\sin\left(\frac{a\cdot\mu_n\cdot t}{R}\right)\right]\cdot\frac{\sin\left(\frac{\mu_n\cdot r}{R}\right)}{r}, \qquad (12)$$

where: a =2,

$$a_n=\frac{2}{R}\cdot\left(1+\frac{1}{\mu_n^2}\right)\cdot\int_0^R\rho\cdot f(\rho)\cdot\sin\left(\frac{\mu_n\cdot\rho}{R}\right)\cdot d\rho, \qquad (13)$$

$$b_n=\frac{2}{a\cdot R}\cdot\left(1+\frac{1}{\mu_n^2}\right)\cdot\int_0^R\rho\cdot F(\rho)\cdot\sin\left(\frac{\mu_n\cdot\rho}{R}\right)\cdot d\rho, \qquad (14)$$

and μ_n are positive roots of the equation $tg(\mu)=\mu$.

2. Calculation of roots μ_n

To determine the roots μ_n we use the graphical representation from figure 2 and MathCAD software [4, 5]:

- insert function $f(x)=\tan(x)-x$;
- choose an approximate value of the solution x;
- the solution is inferred using the function ***root***

$x_j=\mathrm{root}(f(x),x)$, $j=1,2,...$

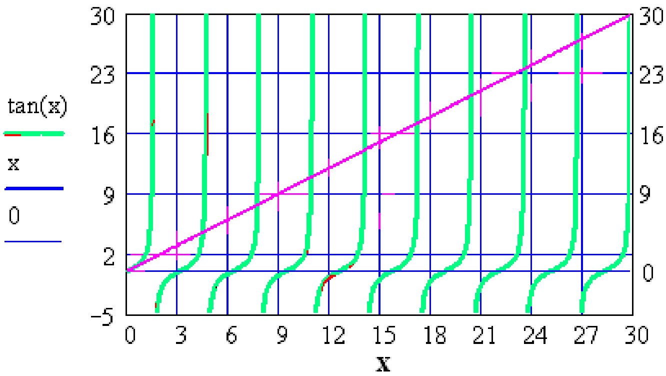

Figure 2 Graphical representation of functions tan(x), x

The first ten positive roots of the equation $\tan(x)-x=0$ are shown in the second column of the Sol 1 matrix:

$$Sol1:=\begin{pmatrix}1 & 0.1365\\ 2 & 4.4934\\ 3 & 7.7253\\ 4 & 10.9041\\ 5 & 14.0662\\ 6 & 17.2208\\ 7 & 20.3713\\ 8 & 23.5195\\ 9 & 26.661\\ 10 & 29.8116\end{pmatrix}$$

3. Analytical solution analysis

We consider $R = 2$, $a = 5$, with conditions (8) as:

$$f(r) := \begin{cases} 10^8 \cdot \sin\left(\frac{\pi}{2} \cdot \frac{10 \cdot r}{R}\right) & \text{if } 0 \le r \le \frac{R}{10} \\ 10^5 & \text{otherwise} \end{cases}$$

and its representation is shown in figure 3 [4, 5]:

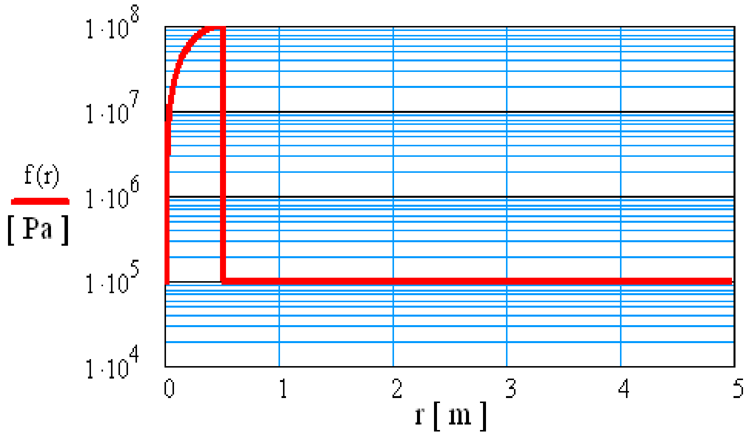

Figure 3 Graphical representation of the initial condition, $p(r,0) = f(r)$

We chose: $F(r) = 10^3$, $a = 2$, then we calculate the value of a_n and b_n, for $n = 1,...,10$, using relations (13) and (14):

$n =$	$a_n =$	$b_n =$
1	$4.678 \cdot 10^6$	$2.433 \cdot 10^4$
2	$1.358 \cdot 10^6$	0.022
3	$2.167 \cdot 10^6$	0.063
4	$2.042 \cdot 10^6$	0.02
5	$3.342 \cdot 10^6$	$4.338 \cdot 10^{-3}$
6	$3.04 \cdot 10^6$	-0.026
7	$3.721 \cdot 10^6$	$-1.455 \cdot 10^{-3}$
8	$3.587 \cdot 10^6$	-0.02
9	$3.254 \cdot 10^6$	-1.097
10	$2.751 \cdot 10^6$	$-4.066 \cdot 10^{-4}$

From formula (12) resulting numerical values of the pressure drop across the radius r,

r [m]	$p(r,0)_1$ [Pa]	$p(r,0.2)_1$ [Pa]	$p(r,0.4)_1$ [Pa]
0	$1.937 \cdot 10^7$	$1.937 \cdot 10^7$	$1.937 \cdot 10^7$
0.05	$1.172 \cdot 10^8$	$3.671 \cdot 10^7$	$3.729 \cdot 10^7$
0.1	$1.148 \cdot 10^8$	$3.652 \cdot 10^7$	$3.673 \cdot 10^7$
0.15	$1.108 \cdot 10^8$	$3.62 \cdot 10^7$	$3.581 \cdot 10^7$
0.2	$1.055 \cdot 10^8$	$3.574 \cdot 10^7$	$3.459 \cdot 10^7$
0.25	$9.909 \cdot 10^7$	$3.515 \cdot 10^7$	$3.312 \cdot 10^7$
0.3	$9.17 \cdot 10^7$	$3.441 \cdot 10^7$	$3.146 \cdot 10^7$
0.35	$8.364 \cdot 10^7$	$3.353 \cdot 10^7$	$2.967 \cdot 10^7$
0.4	$7.517 \cdot 10^7$	$3.252 \cdot 10^7$	$2.784 \cdot 10^7$
0.45	$6.658 \cdot 10^7$	$3.137 \cdot 10^7$	$2.604 \cdot 10^7$
0.5	$5.813 \cdot 10^7$	$3.011 \cdot 10^7$	$2.431 \cdot 10^7$

r [m]	$p(r,0)_1$ [Pa]	$p(r,0.2)_1$ [Pa]	$p(r,0.4)_1$ [Pa]
0.5	$5.813 \cdot 10^7$	$3.011 \cdot 10^7$	$2.431 \cdot 10^7$
1	$1.496 \cdot 10^7$	$1.846 \cdot 10^7$	$1.763 \cdot 10^7$
1.5	$2.129 \cdot 10^7$	$2.183 \cdot 10^7$	$1.765 \cdot 10^7$
2	$1.834 \cdot 10^7$	$1.859 \cdot 10^7$	$2.044 \cdot 10^7$
2.5	$2.011 \cdot 10^7$	$1.744 \cdot 10^7$	$2.149 \cdot 10^7$
3	$1.89 \cdot 10^7$	$2.042 \cdot 10^7$	$1.613 \cdot 10^7$
3.5	$1.98 \cdot 10^7$	$1.896 \cdot 10^7$	$1.938 \cdot 10^7$
4	$1.909 \cdot 10^7$	$1.829 \cdot 10^7$	$1.901 \cdot 10^7$
4.5	$1.968 \cdot 10^7$	$1.807 \cdot 10^7$	$1.96 \cdot 10^7$
5	$1.916 \cdot 10^7$	$2.24 \cdot 10^7$	$2.165 \cdot 10^7$

with graphic as shown in figure 4 [3, 4, 5]:

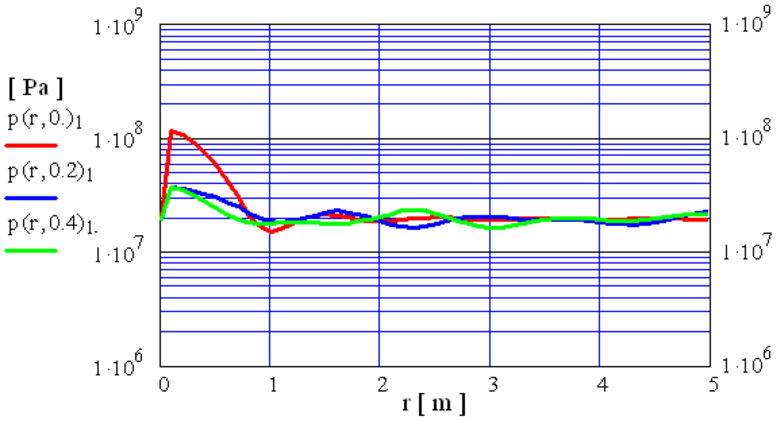

Figure 4 Graphical representation of the solution p(r, t) for three values of time t

4. Conclusion

To highlight the influence of initial conditions on the evolution of the pressure wave we analyze two types of conditions:

1. $F(r) = 10^3$

$$f(r) := \begin{cases} 10^8 \cdot \sin\left(\frac{\pi}{2} \cdot \frac{10 \cdot r}{R}\right) & \text{if } 0 \le r \le \frac{R}{10} \\ 10^5 & \text{otherwise} \end{cases}$$

2. $F1(r1) = 10^6$

$$f1(r1) := \begin{cases} 10^8 \cdot \cos\left(\frac{\pi}{2} \cdot \frac{10 \cdot r1}{R}\right) & \text{if } 0 \le r1 < \frac{R}{10} \\ 10^5 & \text{otherwise} \end{cases}$$

,

with graphic as in shown in figure 5 [3, 5]:

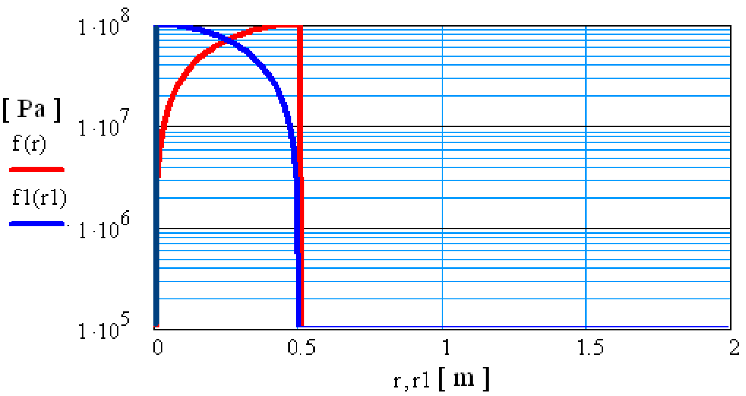

Fig. 5 Representation of initial conditions

Variations in time of pressure wave, corresponding to the two initial conditions are:

r [m]	p(r,0.2)₁ [Pa]	p1(r1,0.2)₁ [Pa]	p(r,0.4)₁ [Pa]	p1(r1,0.4)₁ [Pa]
0	$1.937\cdot10^{7}$	$1.997\cdot10^{7}$	$1.937\cdot10^{7}$	$2.057\cdot10^{7}$
0.05	$3.671\cdot10^{7}$	$3.731\cdot10^{7}$	$3.729\cdot10^{7}$	$3.849\cdot10^{7}$
0.1	$3.652\cdot10^{7}$	$3.712\cdot10^{7}$	$3.673\cdot10^{7}$	$3.793\cdot10^{7}$
0.15	$3.62\cdot10^{7}$	$3.68\cdot10^{7}$	$3.581\cdot10^{7}$	$3.701\cdot10^{7}$
0.2	$3.574\cdot10^{7}$	$3.634\cdot10^{7}$	$3.459\cdot10^{7}$	$3.579\cdot10^{7}$
0.25	$3.515\cdot10^{7}$	$3.575\cdot10^{7}$	$3.312\cdot10^{7}$	$3.432\cdot10^{7}$
0.3	$3.441\cdot10^{7}$	$3.501\cdot10^{7}$	$3.146\cdot10^{7}$	$3.265\cdot10^{7}$
0.35	$3.353\cdot10^{7}$	$3.413\cdot10^{7}$	$2.967\cdot10^{7}$	$3.087\cdot10^{7}$
0.4	$3.252\cdot10^{7}$	$3.311\cdot10^{7}$	$2.784\cdot10^{7}$	$2.904\cdot10^{7}$
0.45	$3.137\cdot10^{7}$	$3.197\cdot10^{7}$	$2.604\cdot10^{7}$	$2.723\cdot10^{7}$
0.5	$3.011\cdot10^{7}$	$3.071\cdot10^{7}$	$2.431\cdot10^{7}$	$2.551\cdot10^{7}$

r [m]	p(r,0.2)₁ [Pa]	p1(r1,0.2)₁ [Pa]	p(r,0.4)₁ [Pa]	p1(r1,0.4)₁ [Pa]
0.5	$3.011\cdot10^{7}$	$3.071\cdot10^{7}$	$2.431\cdot10^{7}$	$2.551\cdot10^{7}$
1	$1.846\cdot10^{7}$	$1.906\cdot10^{7}$	$1.763\cdot10^{7}$	$1.883\cdot10^{7}$
1.5	$2.183\cdot10^{7}$	$2.243\cdot10^{7}$	$1.765\cdot10^{7}$	$1.885\cdot10^{7}$
2	$1.859\cdot10^{7}$	$1.919\cdot10^{7}$	$2.044\cdot10^{7}$	$2.164\cdot10^{7}$
2.5	$1.744\cdot10^{7}$	$1.804\cdot10^{7}$	$2.149\cdot10^{7}$	$2.269\cdot10^{7}$
3	$2.042\cdot10^{7}$	$2.102\cdot10^{7}$	$1.613\cdot10^{7}$	$1.733\cdot10^{7}$
3.5	$1.896\cdot10^{7}$	$1.956\cdot10^{7}$	$1.938\cdot10^{7}$	$2.058\cdot10^{7}$
4	$1.829\cdot10^{7}$	$1.889\cdot10^{7}$	$1.901\cdot10^{7}$	$2.021\cdot10^{7}$
4.5	$1.807\cdot10^{7}$	$1.867\cdot10^{7}$	$1.96\cdot10^{7}$	$2.08\cdot10^{7}$
5	$2.24\cdot10^{7}$	$2.3\cdot10^{7}$	$2.165\cdot10^{7}$	$2.285\cdot10^{7}$

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