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NONLINEAR GENERALIZATIONS OF BIHARI INEQUALITY IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS WITH “MAXIMA”

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Abstract: *This paper deals with some nonlinear integral inequalities that involve the maximum of the unknown scalar function of two variable. The considered inequalities are generalizations of the Bihari inequality. The importance of these integral inequalities is defined by their wide applications in the qualitative investigations of partial differential equations with “maxima” and it is illustrated by some direct applications.*
Keywords: *integral inequalities, maxima, scalar function of two variable, partial differential equations*

1. INTRODUCTION

In the past few years, a number of integral inequalities had been established by many scholars, which are motivated by certain applications such as existence, uniqueness, continuous dependence, comparison, perturbation, boundedness and stability of solutions of differential and integral equations (see, for example, [3], [4], [7], and the references cited therein). Among these integral inequalities, we cite the famous Gronwall inequality and its different generalizations ([5]).

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in the control theory correspond to the maximal deviation of the regulated quantity. Such kind of problems could be adequately modeled by differential equations that contain the maxima operator. Note that equations, involving “maxima” of the unknown function are called differential equations with “maxima”. A. D. Myshkis also points out the necessity to study differential equations with “maxima” in his survey [6].

The purpose of this paper is to establish some new nonlinear integral inequalities in the case when the “maxima” of the unknown scalar function is involved in the integrals. Several cases depending on the additional term to the integrals are considered. These inequalities are mathematical tools in the theory of partial differential equations with “maxima”. Their importance is illustrated on some direct applications to be obtained bounds of the solutions.

2. PRELIMINARY NOTES

Let $h>0$ be a constant and $x_0,\, y_0,\, X,\, Y$ be fixed points such that the inequalities $0 \le x_0 < X \le \infty$ and $0 \le y_0 < Y \le \infty$ are valid.

Definition: We will say that the function $\alpha \in C^1([x_0, X), \mathbb{R}_+)$ is from the class F if it is a nondecreasing function and $\alpha(x) \le x$ for $x \in [x_0, X)$.

Let the functions $\alpha,\, \beta \in \mathrm{F}$ and denote $J = \min\big(\alpha(x_0), \beta(x_0)\big)$. Consider the sets G, Ψ and Λ defined by

$$G = \left\{(x,y) \in \mathbb{R}^2 : x \in [x_0, X),\, y \in [y_0, Y)\right\},$$

$$\Psi = \left\{(x,y) \in \mathbb{R}^2 : x \in [J-h, x_0],\, y \in [y_0, Y)\right\},$$

$$\Lambda = \left\{(x,y) \in \mathbb{R}^2 : x \in [J-h, X),\, y \in [y_0, Y)\right\} = G \cup \Psi.$$

In the investigation of our main results we use the following results:

Lemma 1: ([1]). Assume that $a \ge 0$, $p \ge q \ge 0$ and $p \ne 0$.
Then

$$a^{\frac{q}{p}} \le \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}} \quad \text{for any } K>0.$$

Lemma 2: ([2, Theorem 1]) Let the following conditions be fulfilled:

1. The functions $\alpha_i, \beta_i \in C^1([x_0, X), \mathbb{R}_+)$ are nondecreasing and the inequalities $\alpha_i(x) \le x$, $\beta_i(x) \le x$ hold on $[x_0, X)$ for $i = 1, \ldots, n$.
2. The functions $f_i, g_i \in C([J, X) \times [y_0, Y), \mathbb{R}_+)$ for $i = 1, \ldots, n$.
3. The function $\phi \in C(\Psi, \mathbb{R}_+)$.
4. The function $k \in C(G, (0,\infty))$ is nondecreasing in its both arguments and the inequality $\max\limits_{s \in [J-h, x_0]} \phi(s,y) \le k(x_0, y)$ holds for $y \in [y_0, Y)$.
5. The function $u \in C(\Lambda, \mathbb{R}_+)$ and satisfies the inequalities

$$u(x,y) \le k(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) u(s,t)\, dt\, ds$$
$$+ \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h,s]} u(\xi,t)\, dt\, ds \qquad \text{for } (x,y) \in G, \tag{0.1}$$

$$u(x,y) \le \phi(x,y) \qquad \text{for } (x,y) \in \Psi. \tag{0.2}$$

Then for $(x,y) \in G$ the inequality

$$u(x,y) \le k(x,y) \exp\left(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\, dt\, ds + \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t)\, dt\, ds \right) \tag{0.3}$$

holds.

3. MAIN RESULTS

We will solve some inequalities for unknown function of two variables in the case when the maxima operator is involved in the integrals.

Theorem 1: Let the following conditions be fulfilled:

1. The functions $\alpha, \beta \in \mathrm{F}$.
2. The functions $f, g \in C([J,X) \times [y_0,Y), \mathbb{R}_+)$.
3. The function $\phi \in C(\Psi, \mathbb{R}_+)$.
4. The function $u \in C(\Lambda, \mathbb{R}_+)$ satisfies the inequalities

$$\left(u(x,y)\right)^p \le k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{y_0}^{y} f(s,t) \left(u(s,t)\right)^q dt\, ds$$
$$+ \int_{\beta(x_0)}^{\beta(x)} \int_{y_0}^{y} g(s,t) \left(\max_{\xi \in [s-h,s]} u(\xi,t) \right)^r dt\, ds, \qquad (x,y) \in G, \tag{0.4}$$

$$u(x,y) \le \phi(x,y), \qquad (x,y) \in \Psi, \tag{0.5}$$

where p, q, r and k are constants such that $p \ne 0,\ p \ge q \ge 0,\ p \ge r \ge 0$ and $\mathrm{k} \ge 0.$

Then for $(x,y) \in G$ the inequality

$$u(x,y) \le \left\{ M + \bar{A}(x,y) \exp\left(B(x,y)\right) \right\}^{\frac{1}{p}}. \tag{0.6}$$

holds, where

$$A_1(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{y_0}^{y} f(s,t) \left[\frac{qM}{p} K_1^{\frac{q-p}{p}}(s,t) + \frac{p-q}{p} K_1^{\frac{q}{p}}(s,t) \right] dt\, ds, \tag{0.7}$$

$$A_2(x,y) = \int_{\beta(x_0)}^{\beta(x)} \int_{y_0}^{y} g(s,t) \left[\frac{rM}{p} K_2^{\frac{r-p}{p}}(s,t) + \frac{p-r}{p} K_2^{\frac{r}{p}}(s,t) \right] dt\, ds, \tag{0.8}$$

$$\bar{A}(x,y) = 1 + A_1(x,y) + A_2(x,y) \quad \text{for } (x,y) \in G, \tag{0.9}$$

$$B(x,y) = \frac{q}{p} \int_{\alpha(x_0)}^{\alpha(x)} \int_{y_0}^{y} f(s,t) K_1^{\frac{q-p}{p}}(s,t)\, dt\, ds + \frac{r}{p} \int_{\beta(x_0)}^{\beta(x)} \int_{y_0}^{y} g(s,t) K_2^{\frac{r-p}{p}}(s,t)\, dt\, ds, \tag{0.10}$$

$$M = \max\left(k, \left(\max_{s \in [J-h, x_0]} \phi(s,y) \right)^p \right). \tag{0.11}$$

Proof: Define a function $z : \Lambda \to \mathbb{R}_+$ by the equalities

$$z(x,y)=\begin{cases} M+\int\limits_{\alpha(x_0)}^{\alpha(x)}\int\limits_{y_0}^{y} f(s,t)\big(u(s,t)\big)^q\,dt\,ds \\ \quad +\int\limits_{\beta(x_0)}^{\beta(x)}\int\limits_{y_0}^{y} g(s,t)\left(\max\limits_{\xi\in[s-h,s]} u(\xi,t)\right)^r dt\,ds, & (x,y)\in G \\ M, & (x,y)\in\Psi \end{cases}$$

where the constant M is defined by (1.66).

Note the function $z(x,y)$ is nondecreasing in its both arguments, $z(x_0,y_0)=M$ and the following inequalities hold

$$u(x,y)\le\big(z(x,y)\big)^{\frac{1}{p}},\quad (x,y)\in\Lambda \qquad (0.12)$$

$$\max_{\xi\in[s-h,s]} u(\xi,t)\le\max_{\xi\in[s-h,s]} z^{\frac{1}{p}}(\xi,t)=z^{\frac{1}{p}}(s,t),\quad s\in[\beta(x_0),\beta(X)),\ t\in[y_0,Y). \qquad (0.13)$$

From inequalities (1.59), (1.67), (1.68) and the definition of the function $z(x,y)$ we get for $(x,y)\in G$

$$\begin{aligned} z(x,y)&=M+\int\limits_{\alpha(x_0)}^{\alpha(x)}\int\limits_{y_0}^{y} f(s,t)\big(u(s,t)\big)^q\,dt\,ds+\int\limits_{\beta(x_0)}^{\beta(x)}\int\limits_{y_0}^{y} g(s,t)\left(\max\limits_{\xi\in[s-h,s]} u(\xi,t)\right)^r dt\,ds \\ &\le M+\int\limits_{\alpha(x_0)}^{\alpha(x)}\int\limits_{y_0}^{y} f(s,t)\big(z(s,t)\big)^{\frac{q}{p}}\,dt\,ds+\int\limits_{\beta(x_0)}^{\beta(x)}\int\limits_{y_0}^{y} g(s,t)\big(z(s,t)\big)^{\frac{r}{p}}\,dt\,ds. \end{aligned} \qquad (0.14)$$

Define a function $\overline{Z}:G\to\mathbb{R}_+$ by the equality

$$\overline{Z}(x,y)=\int\limits_{\alpha(x_0)}^{\alpha(x)}\int\limits_{y_0}^{y} f(s,t)\big(z(s,t)\big)^{\frac{q}{p}}\,dt\,ds+\int\limits_{\beta(x_0)}^{\beta(x)}\int\limits_{y_0}^{y} g(s,t)\big(z(s,t)\big)^{\frac{r}{p}}\,dt\,ds.$$

Therefore inequality (1.69) can be written in the form

$$z(x,y)\le M+\overline{Z}(x,y),\quad (x,y)\in G. \qquad (0.15)$$

According to Lemma 1 for any $(x,y)\in G$ there exist functions $K_1(x,y)$ and $K_2(x,y)$ such that

$$\big(z(x,y)\big)^{\frac{q}{p}}\le\big(M+\overline{Z}(x,y)\big)^{\frac{q}{p}}\le\frac{q}{p}K_1^{\frac{q-p}{p}}(x,y)\big(M+\overline{Z}(x,y)\big)+\frac{p-q}{p}K_1^{\frac{q}{p}}(x,y) \qquad (0.16)$$

and

$$\big(z(x,y)\big)^{\frac{r}{p}}\le\big(M+\overline{Z}(x,y)\big)^{\frac{r}{p}}\le\frac{r}{p}K_2^{\frac{r-p}{p}}(x,y)\big(M+\overline{Z}(x,y)\big)+\frac{p-r}{p}K_2^{\frac{r}{p}}(x,y). \qquad (0.17)$$

From (1.71), (1.72) and the definition of the function $\overline{Z}(x,y)$ it follows

$$\begin{aligned} \overline{Z}(x,y)&\le\overline{A}(x,y)+\frac{q}{p}\int\limits_{\alpha(x_0)}^{\alpha(x)}\int\limits_{y_0}^{y} f(s,t)K_1^{\frac{q-p}{p}}(s,t)\overline{Z}(s,t)\,dt\,ds \\ &\quad+\frac{r}{p}\int\limits_{\beta(x_0)}^{\beta(x)}\int\limits_{y_0}^{y} g(s,t)K_2^{\frac{r-p}{p}}(s,t)\overline{Z}(s,t)\,dt\,ds, \end{aligned} \qquad (0.18)$$

where the function $\overline{A}(x,y)$ is defined by equality (1.64). Note the function $\overline{A}(x,y)$ is continuous and nondecreasing in its both arguments for $(x,y)\in G,\ \overline{A}(x_0,y_0)=1.$

Then from inequality (1.73) we get

$$\frac{\bar{Z}(x,y)}{\bar{A}(x,y)} \le 1 + \frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)K_1^{\frac{q-p}{p}}(s,t)\frac{\bar{Z}(s,t)}{\bar{A}(s,t)}dt\,ds$$
$$+\frac{r}{p}\int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)K_2^{\frac{r-p}{p}}(s,t)\frac{\bar{Z}(s,t)}{\bar{A}(s,t)}dt\,ds. \quad (0.19)$$

Define a function $w: G \to \mathbb{R}_+$ by $w(x,y) = \frac{\bar{Z}(x,y)}{\bar{A}(x,y)}$. Then from (1.74) it follows

$$w(x,y) \le 1 + \frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)K_1^{\frac{q-p}{p}}(s,t)w(s,t)dt\,ds$$
$$+\frac{r}{p}\int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)K_2^{\frac{r-p}{p}}(s,t)w(s,t)dt\,ds. \quad (0.20)$$

According to Lemma 2 for $k(x,y) \equiv 1,\ \mathrm{n}=1,\ \max_{\xi\in[s-h,s]} w(\xi,t) \equiv w(s,t),$ where $s \in [J,X),\ t \in [y_0,Y),$ from inequality (1.75) we obtain

$$w(x,y) \le \exp\big(B(x,y)\big), \quad (x,y)\in G \quad (0.21)$$

where the function $B(x,y)$ is defined by equality (1.65).

Inequalities (1.67), (1.70), (1.76) and the definition of the function $w(x,y)$ imply the validity of (1.61).

In the case when the additional term to the integrals is a monotonic function instead of a constant the following result is valid:

Theorem 2: Let the following conditions be fulfilled:

1. The conditions 1, 2 and 3 of Theorem 1 are satisfied.
2. The function $k \in C(G,\mathbb{R}_+)$ is nondecreasing in its both arguments and the inequality $\max_{s\in[J-h,x_0]} \phi(s,y) \le \sqrt[p]{k(x_0,y_0)}$ is valid.
3. The function $u \in C(\Lambda,\mathbb{R}_+)$ satisfies the inequalities

$$\big(u(x,y)\big)^p \le k(x,y) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)\big(u(s,t)\big)^q dt\,ds$$
$$+\int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)\Big(\max_{\xi\in[s-h,s]} u(\xi,t)\Big)^r dt\,ds, \qquad (x,y)\in G, \quad (0.22)$$

$$u(x,y) \le \phi(x,y), \qquad (x,y)\in\Psi, \quad (0.23)$$

where p,q,r are constants such that $p \ne 0,\ \ p \ge q \ge 0$ and $p \ge r \ge 0.$

Then for $(x,y)\in G$ the inequality

$$u(x,y) \le \Big\{k(x,y) + \tilde{A}(x,y)\exp\big(B_1(x,y)\big)\Big\}^{\frac{1}{p}} \quad (0.24)$$

holds, where

$$A_3(x,y) = \int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)\left[\frac{q}{p}K_1^{\frac{q-p}{p}}(s,t)k(s,t) + \frac{p-q}{p}K_1^{\frac{q}{p}}(s,t)\right]dt\,ds, \quad (0.25)$$

$$A_4(x,y) = \int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)\left[\frac{r}{p}K_2^{\frac{r-p}{p}}(s,t)\max_{\xi\in[s-h,s]} k(\xi,t) + \frac{p-r}{p}K_2^{\frac{r}{p}}(s,t)\right]dt\,ds, \quad (0.26)$$

$$\tilde{A}(x,y) = 1 + A_3(x,y) + A_4(x,y) \quad \text{for } (x,y)\in G, \quad (0.27)$$

$$B_1(x,y)=\frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)K_1^{\frac{q-p}{p}}(s,t)dt\,ds+\frac{r}{p}\int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)K_2^{\frac{r-p}{p}}(s,t)dt\,ds. \quad (0.28)$$

Proof: Define a nondecreasing function $z:\Lambda\to\mathbb{R}_+$ by the equalities

$$z(x,y)=\begin{cases}\int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)\left(u(s,t)\right)^q dt\,ds \\ \quad +\int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)\left(\max_{\xi\in[s-h,s]} u(\xi,t)\right)^r dt\,ds, & (x,y)\in G,\\ 0, & (x,y)\in\Psi.\end{cases}$$

From inequality (1.77) and the definition of the function $z(x,y)$ we get

$$u(x,y)\le\left(k(x,y)+z(x,y)\right)^{\frac{1}{p}},\quad (x,y)\in G \quad (0.29)$$

$$\max_{\xi\in[s-h,s]} u(\xi,t)\le\left(\max_{\xi\in[s-h,s]} k(\xi,t)+\max_{\xi\in[s-h,s]} z(\xi,t)\right)^{\frac{1}{p}},\ s\in[\beta(x_0),\beta(X)),\ t\in[y_0,Y). \quad (0.30)$$

According to Lemma 1 for any $(x,y)\in G$ there exist functions $K_1(x,y)$ and $K_2(x,y)$ such that from (1.84) and (1.85) we get

$$\begin{aligned}z(x,y)&=\int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)\left(u(s,t)\right)^q dt\,ds+\int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)\left(\max_{\xi\in[s-h,s]} u(\xi,t)\right)^r dt\,ds\\
&\le\int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)\left[\frac{q}{p}K_1^{\frac{q-p}{p}}(s,t)\left(k(s,t)+z(s,t)\right)+\frac{p-q}{p}K_1^{\frac{q}{p}}(s,t)\right]dt\,ds\\
&\quad+\int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)\left[\frac{r}{p}K_2^{\frac{r-p}{p}}(s,t)\left(\max_{\xi\in[s-h,s]} k(\xi,t)+\max_{\xi\in[s-h,s]} z(\xi,t)\right)+\frac{p-r}{p}K_2^{\frac{r}{p}}(s,t)\right]dt\,ds \quad (0.31)\\
&\le\tilde{A}(x,y)+\frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)K_1^{\frac{q-p}{p}}(s,t)z(s,t)dt\,ds\\
&\quad+\frac{r}{p}\int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)K_2^{\frac{r-p}{p}}(s,t)\max_{\xi\in[s-h,s]} z(\xi,t)dt\,ds,\end{aligned}$$

where the function $\tilde{A}(x,y)$ is defined by equality (1.82).

From the monotonicity of the function $\tilde{A}(x,y)$ we get for $t\in[y_0,Y)$ and $s\in[J,X)$

$$\frac{\max_{\xi\in[s-h,s]} z(\xi,t)}{\tilde{A}(s,t)}\le\frac{\max_{\xi\in[s-h,s]} z(\xi,t)}{\tilde{A}_1(s,t)}\le\max_{\xi\in[s-h,s]}\frac{z(\xi,t)}{\tilde{A}_1(s,t)}=\max_{\xi\in[s-h,s]}\frac{z(\xi,t)}{\tilde{A}_1(\xi,t)}, \quad (0.32)$$

where the continuous nondecreasing function $\tilde{A}_1:\Lambda\to[1,\infty)$ is defined by

$$\tilde{A}_1(x,y)=\begin{cases}\tilde{A}(x,y), & (x,y)\in G,\\ 1, & (x,y)\in\Psi.\end{cases}$$

Define a function $\tilde{w}:G\to\mathbb{R}_+$ by the equality $\tilde{w}(x,y)=\dfrac{z(x,y)}{\tilde{A}_1(x,y)}$. From (1.86) and (1.87) we get

$$\tilde{w}(x,y) \le 1 + \frac{q}{p}\int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y} f(s,t)K_1^{\frac{q-p}{p}}(s,t)\tilde{w}(s,t)\,dt\,ds$$
$$+\frac{r}{p}\int_{\beta(x_0)}^{\beta(x)}\int_{y_0}^{y} g(s,t)K_2^{\frac{r-p}{p}}(s,t)\max_{\xi\in[s-h,s]}\tilde{w}(\xi,t)\,dt\,ds, \qquad (x,y)\in G. \tag{0.33}$$

According to Lemma 2 for $k(x,y)\equiv 1,\ \mathrm{n}=1,$ from inequality (1.88) we get

$$\tilde{w}(x,y) \le \exp\left(B_1(x,y)\right), \quad (x,y)\in G \tag{0.34}$$

where the function $B_1(x,y)$ is defined by equality (1.83)

From inequalities (1.84) and (1.89) the definitions of the functions $\tilde{w}(x,y)$ and $\tilde{A}_1(x,y)$ we obtain the required inequality (1.79).

4. APPLICATIONS

We will apply the solved above inequalities to obtain some qualitative investigations of partial differential equations with “maxima”.

Example: Consider the following partial differential equation with “maxima”

$$u''_{xy} = F\left(x, y, u(x,y), \max_{s\in[\sigma(x),\tau(x)]} u(s,y)\right) \text{ for } x\ge x_0,\ y\ge y_0 \tag{0.35}$$

with the initial and boundary conditions

$$\begin{aligned} &u(x_0,y)=\varphi_1(y) && \text{for } y\in[y_0,Y),\\ &u(x,y_0)=\varphi_2(x) && \text{for } x\in[x_0,X),\\ &u(x,y)=\phi(x,y) && \text{for } (x,y)\in[\tau(x_0)-h,x_0]\times[y_0,Y), \end{aligned} \tag{0.36}$$

where $u\in\mathbb{R}$, $\varphi_1:[y_0,Y)\to\mathbb{R}$, $\varphi_2:[x_0,X)\to\mathbb{R}$, $\phi:[\tau(x_0)-h,x_0]\times[y_0,Y)\to\mathbb{R}$, $F:[x_0,X)\times[y_0,Y)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Theorem 3: (Upper bound). Let the following conditions be fulfilled:

1. The functions $\tau,\sigma\in\mathrm{F}$ and there exists a constant $h>0$ such that $0<\tau(x)-\sigma(x)\le h$ for $x\ge x_0$.
2. The function $F\in C([x_0,X)\times[y_0,Y)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and satisfies for $x\ge x_0,\ y\ge y_0$ and $\gamma,\nu\in\mathbb{R}$, the condition
$$\left|F(x,y,\gamma,\nu)\right| \le Q(x,y)|\gamma|^{\tilde{q}} + R(x,y)|\nu|^{\tilde{r}},$$
where $Q,R\in C(G,\mathbb{R}_+)$ and the constants $\tilde{q}\in[0,1],\ \tilde{r}\in[0,1].$
3. The function $\phi\in C([\tau(x_0)-h,x_0]\times[y_0,Y),\mathbb{R}).$
4. The functions $\varphi_1\in C([y_0,Y),\mathbb{R}),\ \varphi_2\in C([x_0,X),\mathbb{R})$ and the equalities $\varphi_1(y_0)=\varphi_2(x_0),\ \varphi_1(y)=\phi(x_0,y)$ for $y\in[y_0,Y)$ hold.
5. The function $u(x,y)$ is a solution of the initial value problem (1.90), (1.91) which is defined for $(x,y)\in[\tau(x_0)-h,X)\times[y_0,Y).$

Then the solution $u(x,y)$ of the partial differential equation with “maxima” (1.90), (1.91) satisfies for $(x,y)\in G$ the inequality

$$\left|u(x,y)\right| \le \hat{k}(x,y) + \hat{A}(x,y)\exp\left(\hat{B}(x,y)\right), \tag{0.37}$$

where

$$\hat{k}(x,y) = \begin{cases} \left|\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)\right|, & (x,y)\in G\\ \left|\psi(x,y)\right|, & (x,y)\in[\tau(x_0)-h,x_0]\times[y_0,Y) \end{cases} \tag{0.38}$$

$$A_5(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} Q(s,t)\left[\tilde{q}K_1^{\tilde{q}-1}(s,t)\hat{k}(s,t) + (1-\tilde{q})\,K_1^{\tilde{q}}(s,t)\right]dt\,ds, \tag{0.39}$$

$$A_6(x,y)=\int_{x_0}^{x}\int_{y_0}^{y}R(s,t)\left[\tilde{r}K_2^{\tilde{r}-1}(s,t)\max_{\xi\in[s-h,s]}\hat{k}(\xi,t)+(1-\tilde{r})K_2^{\tilde{r}}(s,t)\right]dt\,ds, \qquad (0.40)$$

$$\hat{A}(x,y)=1+A_5(x,y)+A_6(x,y)\quad \text{for } (x,y)\in G, \qquad (0.41)$$

$$\hat{B}(x,y)=\tilde{q}\int_{x_0}^{x}\int_{y_0}^{y}Q(s,t)K_1^{\tilde{q}-1}(s,t)dt\,ds+\tilde{r}\int_{x_0}^{x}\int_{y_0}^{y}R(s,t)K_2^{\tilde{r}-1}(s,t)dt\,ds. \qquad (0.42)$$

Proof: For the function $u(x,y)$ we obtain

$$\begin{aligned}\left|u(x,y)\right|&\le\left|\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)\right|+\int_{x_0}^{x}\int_{y_0}^{y}\left|F\Big(s,t,u(s,t),\max_{\xi\in[\sigma(s),\tau(s)]}u(\xi,t)\Big)\right|dt\,ds\\&\le\left|\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)\right|+\int_{x_0}^{x}\int_{y_0}^{y}Q(s,t)\,|u(s,t)|^{\tilde{q}}\,dt\,ds \qquad (0.43)\\&\quad+\int_{x_0}^{x}\int_{y_0}^{y}R(s,t)\left|\max_{\xi\in[\sigma(s),\tau(s)]}u(\xi,t)\right|^{\tilde{r}}dt\,ds, \qquad (x,y)\in G,\end{aligned}$$

$$\left|u(x,y)\right|=\left|\phi(x,y)\right|, \qquad (x,y)\in[\tau(x_0)-h,x_0]\times[y_0,Y). \qquad (0.44)$$

Change the variable $s=\tau^{-1}(\eta)$ in the second integral of (1.98), use the inequality $\max_{\xi\in[\sigma(x),\tau(x)]}u(\xi,t)\le\max_{\xi\in[\tau(x)-h,\tau(x)]}u(\xi,t)$ for $t\in[y_0,Y)$ and $x\in[x_0,X)$ that follows from condition 1 of Theorem 3 to obtain

$$\begin{aligned}\left|u(x,y)\right|&\le\hat{k}(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}Q(s,t)\left|u(s,t)\right|^{\tilde{q}}dt\,ds\\&\quad+\int_{\tau(x_0)}^{\tau(x)}\int_{y_0}^{y}R(\tau^{-1}(\eta),t)(\tau^{-1}(\eta))'\left|\max_{\xi\in[\eta-h,\eta]}u(\xi,t)\right|^{\tilde{r}}dt\,d\eta, \quad (x,y)\in G.\end{aligned} \qquad (0.45)$$

Note the conditions of Theorem 2 are satisfied for $\alpha(x)\equiv x,\ \beta(x)\equiv\tau(x),\ k(x,y)\equiv\hat{k}(x,y),$ where the function $\hat{k}(x,y)$ is defined by (1.96), $f(x,y)\equiv Q(x,y)$ for $(x,y)\in G,\ g(x,y)\equiv R(\tau^{-1}(x),y)(\tau^{-1}(x))'$ for $x\in[\tau(x_0),X),$ $y\in[y_0,Y),\ p\equiv 1.$

According to Theorem 2 from inequalities (1.100) and (1.99) we obtain the following bound

$$\left|u(x,y)\right|\le\hat{k}(x,y)++\hat{A}(x,y)\exp\left(\hat{B}(x,y)\right), \quad (x,y)\in G \qquad (0.46)$$

where the function $\hat{A}(x,y)$ and $\hat{B}(x,y)$ are defined by equalities (1.96) and (1.97), respectively.

Inequality (1.101) imply the validity of (1.92).

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REFERENCES

[1] W.N. Li, M.A. Han, F.W. Meng , Some new delay integral inequalities and their applications, *J. Comput. Appl. Math.*, **180**, (2005), 191-200.
[2] S.G. Hristova, K.V. Stefanova, Two-dimensional integral inequalities and applications to partial differential equations with "maxima", *IEJPAM*, Vol. 1, No. 1, (2010), 11-28.
[3] D.D. Bainov, P.S. Simeonov, *Integral inequalities and applications*, Kluwer, (1989).
[4] V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities*, Volume 1, Academic Press, New York, (1969).
[5] Y.H. Kim, On some new integral inequalities for functions in one and two variables, *Acta Mathematica Sinica, English Series*, **21**, No. 2, (2005), 423-434.
[6] A.D. Myshkis, On some problems of the theory of differential equations with deviating argument, *Russian Math. Surveys*, **32**, No. 2, (1977), 181--210 (in Russian).
[7] B.G. Pachpatte, *Inequalities for Differential and Integral Equations*, **197**, Mathematics in Science and Engineering, Academic Press Inc., San Diego, CA, (1998).