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INTEGRO-SUMMATION INEQUALITIES AND APPLICATIONS TO IMPULSIVE DIFFERENTIAL EQUATIONS WITH "SUPREMUM"

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Abstract: *This paper deals with some qualitative properties of the solutions of impulsive differential equations with "supremum". Initially several integral inequalities for piecewise continuous function that involve the supremum of the unknown function over a past time interval are solved. These inequalities are generalizations of the classical integral inequality of Gronwall-Bellman. They are a mathematical tool for studying boundedness of the solutions of impulsive differential equations with "supremum".*
Keywords: *integral inequalities, supremum, boundedness, impulses*

1. INTRODUCTION

Many problems in the control theory correspond to the maximal deviation of the regulated quantity. Such kind of real world problems are adequately modeled by differential equations with "maxima" ([14]). In connection with many possible applications it is absolutely necessary to be developed qualitative theory of differential equations with "maxima" (see the monograph [5] and papers [4], [6], [11]). One of the main mathematical tools, employed successfully for studying existence, uniqueness, continuous dependence, comparison, perturbation, boundedness, and stability of solutions of differential and integral equations is the method of integral inequalities ([1], [3], [13], [14]). Note that in the case when the regulated quantity is perturbed instanstenuously, then it is better to be modeled by a piecewise continuous function and differential equations with impulses. In this case, if the process depends significantly by the maximum/supremum value over a past time interval, the corresponding equations are called impulsive ordinary differential equations with "supremum" ([10]). The development of the theory of these types of equations requires solving of integro-summation inequalities that involve the supremum of the unknown piecewise continuous function.

The main purpose of the paper is solving various types of linear integro-summation inequalities containing supremum of the unknown function. The results are applied to be obtained some bounds for impulsive ordinary differential equations with "supremum".

2. PRELIMINARY NOTES

Let $h>0$ be a constant, $t_0 \geq 0$ be a fixed point such that $t_0 < T \leq \infty$ and the points $t_k \geq 0$ where $k=1,2,\dots$ be such that $t_k < t_{k+1}$ and $\lim_{k\to\infty} t_k = \infty.$

Definition 1: We will say that the function $\alpha \in C^1([t_0,T),\mathbb{R}_+)$ is from the class F if it is a nondecreasing function and $\alpha(t) \leq t$ for $t \in [t_0,T).$

Denote $J = \min_{1\leq j\leq n} \alpha_j(t_0).$

Definition 2: Denote by $PC(\Lambda,\mathbb{R})$ $(\Lambda \subset \mathbb{R})$ the set of all function $u:\Lambda \to \mathbb{R}$ which are piecewise continuous, i.e. there exist limits $\lim_{t\downarrow t_k} u(t) = u(t_k+0) < \infty$ and $\lim_{t\uparrow t_k} u(t) = u(t_k-0) = u(t_k) < \infty$, $t_k \in \Lambda.$

Definition 3: Denote by $PC^1(\Lambda,\mathbb{R})$ $(\Lambda \subset \mathbb{R})$ the set of all functions $u \in PC(\Lambda,\mathbb{R})$ that are continuously differentiable for all $t \in \Lambda$ in which the function is continuous and there exist left derivatives at the points of discontinuity.

In the proof of our main results will use the following lemma:

Lemma 1: ([2, Theorem 1.4.1]). Assume that:

1. The sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \dots$, with $\lim_{k\to\infty} t_k = \infty.$
2. The function $m \in PC^1(\mathbb{R}_+,\mathbb{R})$ and $m(t)$ is left-continuous at $t_k, k=1,2, \dots$.
3. For $k=1,\ 2,\ \dots\ ,\ t \geq t_0$

$$m'(t) \leq p(t)m(t) + q(t), \qquad t \neq t_k,$$

$$m(t_k^+) \leq d_k m(t_k) + b_k,$$

where $q,\ p \in PC(\mathbb{R}_+,\mathbb{R}),\ \ b_k = const$ and $d_k = const \geq 0.$

Then

$$m(t) \le m(t_0)\left(\prod_{t_0<t_k<t} d_k\right)\exp\left(\int_{t_0}^{t} p(s)ds\right)+\int_{t_0}^{t}\left(\prod_{s<t_k<t} d_k\right)\exp\left(\int_{s}^{t} p(\sigma)d\sigma\right)q(s)ds$$

$$+\sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} d_j\right)\exp\left(\int_{t_k}^{t} p(s)ds\right)b_k, \quad t \ge t_0.$$

3. MAIN RESULTS

We will establish some new impulsive integral inequalities in the case when the "supremum" of the unknown function is involved in the integrals.

Now we will consider the case when the additional term to the integrals is a constant.

Theorem 1: Let the following conditions be fulfilled:

1. The functions $\alpha \in F$ and $a, b \in C([\alpha(t_0),T),\mathbb{R}_+)$.

2. The function $\phi \in C([\alpha(t_0)-h,t_0],\mathbb{R}_+)$.

3. The function $u \in PC([\alpha(t_0)-h,T),\mathbb{R}_+)$ satisfies the following inequalities

$$u(t) \le \gamma + \sum_{t_0<t_i<t} \beta_i u(t_i) + \int_{\alpha(t_0)}^{\alpha(t)}\left[a(s)u(s)+b(s)\sup_{\xi\in[s-h,s]} u(\xi)\right]ds \text{ for } t\in[t_0,T), \quad (0.1)$$

$$u(t) \le \phi(t) \qquad \text{for } t\in[\alpha(t_0)-h,t_0], \quad (0.2)$$

where $\beta_0 \equiv 0,\ \beta_i \ge 0\ (i=1,2,\dots)$ are constants and $0 \le \gamma \le \max_{s\in[\alpha(t_0)-h,t_0]} \phi(s) = M.$

Then for $t\in[t_0,T)$ the inequality

$$u(t) \le M\left(\prod_{t_0<t_i<t}(1+\beta_i)\right)\exp\left(\int_{\alpha(t_0)}^{\alpha(t)}\left[a(s)+b(s)\right]ds\right) \quad (0.3)$$

holds.

Proof: Define a function $z \in PC([\alpha(t_0)-h,T),\mathbb{R}_+)$ by the equalities

$$z(t)=\begin{cases} M+\sum_{t_0<t_i<t}\beta_i u(t_i) \\ \quad +\int_{\alpha(t_0)}^{\alpha(t)}\left[a(s)u(s)+b(s)\sup_{\xi\in[s-h,s]}u(\xi)\right]ds, & t\in[t_0,T) \\ M, & t\in[\alpha(t_0)-h,t_0]. \end{cases}$$

From the definition of $z(t)$ and inequalities (1.16), (1.17) we get

$$u(t) \le z(t), \quad t\in[\alpha(t_0)-h,T) \quad (0.4)$$

$$\sup_{\xi\in[s-h,s]} u(\xi) \le \sup_{\xi\in[s-h,s]} z(\xi) = z(s), \quad s\in[\alpha(t_0),T). \quad (0.5)$$

Let $t\in[t_0,t_1].$ From (1.19), (1.20) and the definition of $z(t)$ it follows

$$z(t) \le M + \int_{\alpha(t_0)}^{\alpha(t)}\left[a(s)+b(s)\right]z(s)ds, \quad t\in[t_0,t_1]. \quad (0.6)$$

Apply the Gronwall-Bellman inequality to (1.21) and obtain

$$z(t) \le M\exp\left(\int_{\alpha(t_0)}^{\alpha(t)}[a(s)+b(s)]ds\right), \quad t\in[t_0,t_1]. \quad (0.7)$$

Therefore, from (1.19) and (1.22) follows the validity of the required inequality (1.18) for $t\in[t_0,t_1].$

We will use mathematical induction to prove the validity of inequality (1.18) for $t \in [t_0, T)$. Assume that inequality (1.18) is true for a natural number $k > 1$, i.e. it holds for $t \in [t_0, t_k]$.

Let $t \in (t_k, t_{k+1}]$. Then from the definition of the function $z(t)$, inequalities (1.19) and (1.20) we get

$$z(t) \le M + \sum_{i=1}^{k} \beta_i z(t_i) + \sum_{i=1}^{k} \int_{\alpha(t_{i-1})}^{\alpha(t_i)} \big[a(s) + b(s)\big] z(s)ds + \int_{\alpha(t_\kappa)}^{\alpha(t)} \big[a(s) + b(s)\big] z(s)ds. \quad (0.8)$$

Denote by $Z_k = M + \sum_{i=1}^{k} \beta_i z(t_i) + \sum_{i=1}^{k} \int_{\alpha(t_{i-1})}^{\alpha(t_i)} \big[a(s) + b(s)\big] z(s)ds.$ Using some transformations and the assumption that inequality (1.18) holds for $t \in [t_0, t_k]$ it follows

$$\begin{aligned} Z_k &\le M \left\{ 1 + \sum_{i=1}^{k} \beta_i \left(\prod_{j=1}^{i-1} \big(1+\beta_j\big) \right) \exp\left(\int_{\alpha(t_0)}^{\alpha(t_i)} [a(s)+b(s)]ds \right) + \sum_{i=1}^{k} \left(\prod_{j=1}^{i-1} \big(1+\beta_j\big) \right) \times \right. \\ &\quad \left. \times \int_{\alpha(t_{i-1})}^{\alpha(t_i)} \big[a(s)+b(s)\big] \exp\left(\int_{\alpha(t_0)}^{\alpha(s)} \big[a(\mu)+b(\mu)\big] d\mu \right) ds \right\} \\ &\le M \left\{ 1 + \sum_{i=1}^{k} \beta_i \left(\prod_{j=1}^{i-1} \big(1+\beta_j\big) \right) \exp\left(\int_{\alpha(t_0)}^{\alpha(t_i)} [a(s)+b(s)]ds \right) + \sum_{i=1}^{k} \left(\prod_{j=1}^{i-1} \big(1+\beta_j\big) \right) \times \right. \\ &\quad \left. \times \left[\exp\left(\int_{\alpha(t_0)}^{\alpha(t_i)} [a(s)+b(s)]ds \right) - \exp\left(\int_{\alpha(t_0)}^{\alpha(t_{i-1})} [a(s)+b(s)]ds \right) \right] \right\}. \end{aligned} \quad (0.9)$$

Then we get

$$Z_k \le M \left(\prod_{i=1}^{k} \big(1+\beta_i\big) \right) \exp\left(\int_{\alpha(t_0)}^{\alpha(t_k)} \big[a(s)+b(s)\big] ds \right). \quad (0.10)$$

Use the bound (1.25) for Z_k, from inequality (1.23) according to the Gronwall-Bellman inequality we obtain

$$\begin{aligned} z(t) &\le Z_k \exp\left(\int_{\alpha(t_k)}^{\alpha(t)} \big[a(s)+b(s)\big] ds \right) \\ &\le M \left(\prod_{t_0<t_i<t} \big(1+\beta_i\big) \right) \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} \big[a(s)+b(s)\big] ds \right), \quad t \in (t_k, t_{k+1}]. \end{aligned} \quad (0.11)$$

Therefore, inequalities (1.19) and (1.26) imply the validity of the required inequality (1.18) for $t \in (t_k, t_{k+1}]$.

Thus, the inequality (1.18) is valid for $t \in [t_0, T)$.

In the case when in the right part of the inequality there is a monotonic function instead of a constant, the following result is valid:

Theorem 2: Let the following conditions be fulfilled:

1. The conditions 1 and 2 of Theorem 1 are satisfied.
2. The function $k \in C([t_0, T), (0, \infty))$ is nondecreasing and the inequality $M = \max_{s \in [\alpha(t_0)-h, t_0]} \phi(s) \le k(t_0)$ is valid.
3. The function $u \in PC([\alpha(t_0) - h, T), \mathbb{R}_+)$ satisfies the following inequalities

$$u(t) \le k(t) + \sum_{t_0<t_i<t} \beta_i u(t_i) + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)u(s) + b(s) \sup_{\xi\in[s-h,s]} u(\xi) \right] ds, \quad t \in [t_0, T) \qquad (0.12)$$

$$u(t) \le \phi(t), \qquad t \in [\alpha(t_0) - h, t_0], \qquad (0.13)$$

where $\beta_0 \equiv 0, \beta_i \ge 0 \ (i = 1, \ 2, \ \ldots)$ are constants.

Then for $t \in [t_0, T)$ the inequality

$$u(t) \le M \frac{k(t)}{k(t_0)} \left(\prod_{t_0<t_i<t} (1+\beta_i) \right) \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} [a(s) + b(s)] ds \right) \qquad (0.14)$$

holds.

Proof: From inequalities (1.27) we get

$$\frac{u(t)}{k(t)} \le 1 + \sum_{t_0<t_i<t} \beta_i \frac{u(t_i)}{k(t)} + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)\frac{u(s)}{k(t)} + b(s) \frac{\sup_{\xi\in[s-h,s]} u(\xi)}{k(t)} \right] ds, \quad t \in [t_0, T). \qquad (0.15)$$

Use the monotonicity of the function $k(t)$ to obtain

$$\frac{\sup_{\xi\in[s-h,s]} u(\xi)}{k(t)} \le \frac{\sup_{\xi\in[s-h,s]} u(\xi)}{\tilde{k}(s)} = \sup_{\xi\in[s-h,s]} \frac{u(\xi)}{\tilde{k}(s)} \le \sup_{\xi\in[s-h,s]} \frac{u(\xi)}{\tilde{k}(\xi)}, \quad s \in [\alpha(t_0), T)$$

where the continuous nondecreasing function $\tilde{k} : [\alpha(t_0) - h, T) \to \mathbb{R}_+$ is defined by

$$\tilde{k}(t) = \begin{cases} k(t) & \text{for } t \in [t_0, T) \\ k(t_0) & \text{for } t \in [\alpha(t_0) - h, t_0]. \end{cases}$$

Define a function $z \in PC([\alpha(t_0) - h, T), \mathbb{R}_+)$ by $z(t) = \dfrac{u(t)}{\tilde{k}(t)}$. Then

$$z(t) \le 1 + \sum_{t_0<t_i<t} \beta_i z(t_i) + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)z(s) + b(s) \sup_{\xi\in[s-h,s]} z(\xi) \right] ds, \quad t \in [t_0, T) \qquad (0.16)$$

$$z(t) = \frac{u(t)}{k(t_0)} \le \frac{\phi(t)}{k(t_0)}, \qquad t \in [\alpha(t_0) - h, t_0]. \qquad (0.17)$$

From inequalities (1.31) and (1.32), and the definition of the function $z(t)$ according to Theorem 1 we get for $t \in [t_0, T)$ the required inequality (1.29).

In the case when the unknown function is on a power we obtain the following result:

Theorem 3: Let the following conditions be fulfilled:

1. The conditions 1 and 2 of Theorem 1 are satisfied.
2. The function $u \in PC([\alpha(t_0) - h, T), \mathbb{R}_+)$ satisfies the following inequalities

$$\begin{aligned} (u(t))^p \le \gamma^p &+ \sum_{t_0<t_i<t} \beta_i u^p(t_i) \\ &+ p \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)u(s) + b(s) \sup_{\xi\in[s-h,s]} u(\xi) \right] ds, \quad t \in [t_0, T) \end{aligned} \qquad (0.18)$$

$$u(t) \le \phi(t), \qquad t \in [\alpha(t_0) - h, t_0], \qquad (0.19)$$

where the constants $\beta_i \ge 0 \ (i = 1, \ 2, \ \ldots), \gamma \ge 0,$ and $p \in \mathbb{R}, \ p \ne 1.$

Then for $t \in [t_0, T)$ the inequality

$$u(t) \le \left(\prod_{t_0 < t_i < t} (1+\beta_i)^{\frac{1}{p}} \right) \left\{ \tilde{M}^{\frac{p-1}{p}} + (p-1) \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s) + b(s) \right] ds \right\}^{\frac{1}{p-1}} \tag{0.20}$$

holds, where

$$\tilde{M} = \max\left(\gamma^p, \left(\max_{s \in [\alpha(t_0)-h, t_0]} \phi(s) \right)^p \right). \tag{0.21}$$

Proof: Define a function $z \in PC([\alpha(t_0)-h,T),[\tilde{M},\infty))$ by the equalities

$$z(t) = \begin{cases} \tilde{M} + \sum_{t_0 < t_i < t} \beta_i u^p(t_i) + p \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)u(s) + b(s) \sup_{\xi \in [s-h,s]} u(\xi) \right] ds, & t \in [t_0, T) \\ \tilde{M}, & t \in [\alpha(t_0)-h, t_0] \end{cases}$$

where the constant $\tilde{M}$ is defined by (1.36).

From the definitions of $z(t)$ it follows

$$u(t) \le z^{\frac{1}{p}}(t), \qquad t \in [\alpha(t_0)-h, T) \tag{0.22}$$

$$\sup_{\xi \in [s-h,s]} u(\xi) \le \sup_{\xi \in [s-h,s]} z^{\frac{1}{p}}(\xi) = z^{\frac{1}{p}}(s), \qquad s \in [\alpha(t_0), T). \tag{0.23}$$

Then from (1.37), (1.38), and the definition of the function $z(t)$ we get

$$z(t) \le \tilde{M} + \sum_{t_0 < t_i < t} \beta_i z(t_i) + p \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s) + b(s) \right] z^{\frac{1}{p}}(s) ds, \quad t \in [t_0, T). \tag{0.24}$$

Set $v : [t_0, T) \to [\tilde{M}, \infty)$ to be right side of (1.39). Note that $z(t) \le v(t),\ t \in [t_0, T).$ Differentiate $v(t)$, use condition 1 of Theorem 1 and obtain

$$\begin{aligned} v'(t) &= p\left[a(\alpha(t)) + b(\alpha(t)) \right] \left(\alpha(t) \right)' z^{\frac{1}{p}}(\alpha(t)) \\ &\le p\left[a(\alpha(t)) + b(\alpha(t)) \right] \left(\alpha(t) \right)' v^{\frac{1}{p}}(t), \end{aligned} \tag{0.25}$$

$$\frac{v'(t)}{p\, v^{\frac{1}{p}}(t)} \le \left[a(\alpha(t)) + b(\alpha(t)) \right] \left(\alpha(t) \right)'. \tag{0.26}$$

For $i = 1, 2, \dots$ by the definition of the function $v(t)$ we get

$$v(t_i^+) = v(t_i) + \beta_i z(t_i) \le (1+\beta_i) v(t_i). \tag{0.27}$$

Set $V(t) = v^{\frac{p-1}{p}}(t)$ for $t \in [t_0, T),$ multiply the both sides of (1.41) by $p-1 > 0,$ use (1.42) and obtain

$$\begin{aligned} &V'(t) \le (p-1)\left[a(\alpha(t)) + b(\alpha(t)) \right] \left(\alpha(t) \right)', \quad t \ne t_i,\ t \in [t_0, T) \\ &V\left(t_i^+\right) \le (1+\beta_i)^{\frac{p-1}{p}} V(t_i), \qquad i = 1, 2, \dots \end{aligned} \tag{0.28}$$

From (1.43) and equalities $V(t_0) = v^{\frac{p-1}{p}}(t_0) = \tilde{M}^{\frac{p-1}{p}}$ according to Lemma 1 we get

$$V(t)\le\left(\prod_{t_0<t_i<t}(1+\beta_i)^{\frac{p-1}{p}}\right)\left\{\tilde{M}^{\frac{p-1}{p}}+(p-1)\int_{\alpha(t_0)}^{\alpha(t)}\big[a(s)+b(s)\big]ds\right\},\quad t\in[t_0,T). \tag{0.29}$$

Inequalities (1.44) and $u(t)\le z^{\frac{1}{p}}(t)\le\left(v^{\frac{p-1}{p}}(t)\right)^{\frac{1}{p-1}}=\big(V(t)\big)^{\frac{1}{p-1}}$ imply the validity of (1.35).

Theorem 4: Let the following conditions be fulfilled:

1. The functions $\alpha_j\in F$ and $a_j,b_j\in C([J,T),\mathbb{R}_+)$ for $j=1,\ 2,\ \dots,n.$
2. The function $\phi\in C([J-h,t_0],\mathbb{R}_+).$
3. The function $u\in PC([J-h,T),\mathbb{R}_+)$ satisfies the following inequalities

$$\begin{aligned}\big(u(t)\big)^p\le{}&\gamma^p+\sum_{t_0<t_i<t}\beta_i u^p(t_i)\\&+p\sum_{j=1}^{n}\int_{\alpha_j(t_0)}^{\alpha_j(t)}\big[a_j(s)u(s)+b_j(s)\sup_{\xi\in[s-h,s]}u(\xi)\big]ds,\quad t\in[t_0,T)\end{aligned} \tag{0.30}$$

$$u(t)\le\phi(t),\qquad t\in[J-h,t_0] \tag{0.31}$$

where $\beta_i\ (i=1,2,\dots),\gamma\ge 0,$ and $p\in\mathbb{R},\ p\ne 1$ are constants.

Then for $t\in[t_0,T)$ the inequality

$$u(t)\le\left(\prod_{t_0<t_i<t}(1+\beta_i)^{\frac{1}{p}}\right)\left\{\tilde{M}^{\frac{p-1}{p}}+(p-1)\sum_{j=1}^{n}\int_{\alpha_j(t_0)}^{\alpha_j(t)}\big[a_j(s)+b_j(s)\big]ds\right\}^{\frac{1}{p-1}} \tag{0.32}$$

holds, where

$$\tilde{M}_1=\max\Big(\gamma^p,\big(\max_{s\in[J-h,t_0]}\phi(s)\big)^p\Big). \tag{0.33}$$

Proof: The proof of Theorem 4 is similar to proof of Theorem 3 and we omit it.

4. Applications

We will apply some of the solved above integral inequalities to study boundedness of the solutions of impulsive differential equations with “supremum”.

Let following condition be satisfied:

H1. The functions $\sigma,\tau\in C^1([t_0,T),\mathbb{R}_+)$ are nondecreasing, $\tau(t)\le t$ for $t\in[t_0,T)$ and $0\le\tau(t)-\sigma(t)\le h,$ $t\in[t_0,T).$

Denote by $Z(t_0,T)$ the set of all natural numbers k such that $t_k\in(t_0,T).$

Example: Consider the following impulsive differential equation with “supremum”

$$x^{p-1}x'=f\Big(t,\,x(t),\sup_{s\in[\sigma(t),\tau(t)]}x(s)\Big),\quad t\in[t_0,T),\ t\ne t_k,\ k=1,\ 2,\ \dots \tag{0.34}$$

with the jump condition

$$x(t_k+0)-x(t_k-0)=\beta_k\,x(t_k-0),\qquad \text{for } i\in Z(t_0,T), \tag{0.35}$$

and the initial condition

$$x(t)=\varphi(t),\qquad t\in[\tau(t_0)-h,t_0] \tag{0.36}$$

where $x\in\mathbb{R},\ \varphi:[\tau(t_0)-h,t_0]\to\mathbb{R},\ f:[t_0,T)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},\ \beta_k$ are constants, $k=1,\ 2,\ \dots,$ and $p>1$ is a natural number.

Denote by $x(t)=x(t;t_0,\phi)$ the solution of the initial value problem (1.49)-(1.51) and assume that $x(t)\in PC([\tau(t_0)-h,T),\mathbb{R}).$

Theorem 5: (*Upper bound*). Let the following conditions be fulfilled:
1. The condition (H1) is satisfied.

2. The function $f \in C([t_0,T)\times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and for $t\in[t_0,T)$ and $u,v\in\mathbb{R}$, the inequality $\left| f(t,u,v)\right| \le A(t)\left|u\right| + B(t)\left|v\right|$ holds, where the functions $A, B \in C([t_0,T),\mathbb{R}_+)$.
3. The function $\varphi \in C([\tau(t_0)-h,t_0],\mathbb{R})$.

Then the solution $x(t)$ of the initial value problem (1.49)-(1.51) satisfies for $t\in[t_0,T)$ the inequality

$$|x(t)| \le \left(\prod_{t_0<t_i<t}\left(1+|\beta_i^p|\right)^{\frac{1}{p}}\right)\left\{\hat{M}^{\frac{p-1}{p}} + (p-1)\int_{t_0}^{t}\left[A(s)+B(s)\right]ds\right\}^{\frac{1}{p-1}} \tag{0.37}$$

where $\hat{M} = \max\limits_{s\in[\tau(t_0)-h,t_0]} \left|\varphi(s)\right|^p$.

Proof: The function $x(t)$ satisfies the following integral equation

$$\left(x(t)\right)^p = \left(\varphi(t_0)\right)^p + \sum_{t_0<t_k<t} \beta_k^p\, x^p(t_k) + p\int_{t_0}^{t} f\Big(s, x(s), \sup_{\xi\in[\sigma(s),\tau(s)]} x(\xi)\Big)ds, \quad t\in[t_0,T)$$

$$x(t) = \varphi(t), \qquad t\in[\tau(t_0)-h,t_0].$$

According to condition 2 of Theorem 5 we obtain

$$\begin{aligned}\left|x(t)\right|^p &\le \left|\varphi(t_0)\right|^p + \sum_{t_0<t_k<t}\left|\beta_k^p\right| |x(t_k)|^p + p\int_{t_0}^{t}\Big| f\Big(s, x(s), \sup_{\xi\in[\sigma(s),\tau(s)]} x(\xi)\Big)\Big| ds \\ &\le \left|\varphi(t_0)\right|^p + \sum_{t_0<t_k<t}\left|\beta_k^p\right| |x(t_k)|^p \\ &\quad + p\int_{t_0}^{t} A(s)\left|x(s)\right|ds + p\int_{t_0}^{t} B(s)\sup_{\xi\in[\sigma(s),\tau(s)]}\left|x(\xi)\right|ds, \quad t\in[t_0,T)\end{aligned} \tag{0.38}$$

$$\left|x(t)\right| = \left|\varphi(t)\right|, \qquad t\in[\tau(t_0)-h,t_0]. \tag{0.39}$$

Change the variable $s=\tau^{-1}(\eta)$ in the second integral of (1.53), use the inequality $\sup\limits_{\xi\in[\sigma(s),\tau(s)]}|x(\xi)| \le \sup\limits_{\xi\in[\tau(s)-h,\tau(s)]}|x(\xi)|$ for $s\in[t_0,T)$ that follows from condition 1 of Theorem 5 and obtain

$$\begin{aligned}|x(t)|^p &\le |\varphi(t_0)|^p + \sum_{t_0<t_k<t}\left|\beta_k^p\right| |x(t_k)|^p + p\int_{t_0}^{t} A(s)|x(s)|ds \\ &\quad + p\int_{\tau(t_0)}^{\tau(t)} B(\tau^{-1}(\eta))(\tau^{-1}(\eta))' \sup_{\xi\in[\eta-h,\eta]}\left|x(\xi)\right| d\eta.\end{aligned} \tag{0.40}$$

Note the conditions of Theorem 4 are satisfied for $u(t) = |x(t)|$, $\gamma = |\varphi(t_0)|$, $n=2$, $\alpha_1(t)\equiv t$, $\alpha_2(t)\equiv\tau(t)$, $b_1(t)\equiv 0$, $b_2(t)\equiv B(\tau^{-1}(\eta))(\tau^{-1}(\eta))'$ for $t\in[\tau(t_0),T)$, $a_1(t)\equiv A(t)$, $a_2(t)\equiv 0$.

According to Theorem 4 from (1.54) and (1.55) we obtain the required inequality (1.52).

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