"Mircea cel Batran" Naval Academy Scientific Bulletin, Volume XIV – 2011 – Issue 2
Published by "Mircea cel Batran" Naval Academy Press, Constanta, Romania

PLANNING OF MANIPULATOR TRAJECTORIES

Octavian TĂRĂBUȚĂ[1]
[1] Commander, Ph.D., assistant professor, "Mircea cel Batran" Naval Academy

***Abstract**: In this paper we deal with the formalism of describing the desired manipulator motion as sequences of points in space (position and orientation of the manipulator) through which the manipulator must pass, as well as the space curve that it traverses. The space curve that the manipulator hand moves along from the initial location (position and orientation) to the final location is called the path.*
Key words: *Manipulator, path, trajectory, joint interpolated trajectories.*

1. INTRODUCTION

Before moving a robot arm, it is of considerable interest to know whether there are any obstacles present in its' path (obstacle constraint) and whether the manipulator hand must traverse a specified path (path constraint). We are interested in developing suitable formalisms for defining and describing the desired motions of the manipulator hand between the path endpoints. Trajectory planning schemes generally "interpolate" or "approximate" the desired path by a class of polynomial functions and generates a sequence of time-based "control set points" for the control of the manipulator from the initial location to its destination. Path endpoints can be specified either in joint coordinates or in cartesian coordinates.

However, they are usually specified in cartesian coordinates because it is easier to visualize the correct end-effector configurations in cartesian coordinates than in joint coordinates.

2. GENERAL CONSIDERATIONS ON TRAJECTORY PLANNING

Trajectory planning can be conducted either in the joint variable space or in the Cartesian space. For joint-variable space planning, the time history of all joint variables and their first two time derivatives are planned to describe the desired motion manipulator. For Cartesian space planning, the time history of the manipulator hand's position, velocity, and acceleration are planned and the corresponding joint positions, velocities, and accelerations are derived from the hand information. Planning in the joint-variable space has three advantages- (1) the trajectory is planned directly in terms of the controlled variables during motion, (2) the trajectory planning can be done in near real time, and (3) the joint trajectories are easier to plan.

In general, the basic algorithm for generating joint trajectory set points is quite simple:

t = t0
loop: Wait for the next control interval;
t = t +Δt
h(*t*)= where the manipulator joint position should be at time t;
If $t = t_f$, then exit;
Go to loop;
where Δt is the control sampling period for the manipulator.

From the above algorithm, we see that the computation consists of a trajectory function (or trajectory planner) h(t) which must be updated in every control interval.

Thus, one seventh-degree polynomial for each joint variable connecting the initial and final positions would suffice, as would two quartic and one cubic (4-3-4) trajectory segments, two cubics and one quintic (3-5-3) trajectory segments,or five cubic (3-3-3-3-3) trajectory segments. The (4-3-4) trajectory segments will be discussed further in the next section.

To servo a manipulator, it is required that its robot arm's configuration at both the initial and final locations must be specified before the motion trajectory is planned.

One approach is to specify a seventh-degree polynomial for each joint *i*:

(1) $$q_i = a_7 t^7 + a_6 t^6 + a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0,$$

where the unknown coefficients aj can be determined from the known positions and continuity conditions. However, the use of such a high-degree polynomial to interpolate the given knot points not to be satisfactory. It is difficult to find its extrema and it tends to have extraneous motion. An alternative approach is to split entire joint trajectory into several trajectory segments so that different interpolating polynomials of a lower degree can be used to interpolate in each trajectory segment. There are different ways a joint trajectory can be split, and each method possesses different properties. The most common methods are the following:

4-3-4 *Trajectory.* Each joint has the following three trajectory segments: the first segment is a fourth-degree polynomial specifying the trajectory from the initial position to the lift-off position. The second trajectory segment (or midtrajectory segment) is a third-degree polynomial specifying the trajectory from the lift-off position to the set-down position. The last trajectory segment is a fourth-degree polynomial specifying the trajectory from the set-down position to the final position.

3-5-3 *Trajectory.* Same as 4-3-4 trajectory, but uses polynomials of differed degrees for each segment: a third-degree polynomial for the first segment a fifth-degree polynomial for the second segment, and a third-degree polynomial for the last segment.

5-Cubic Trajectory. Cubic spline functions of third-degree polynomials for five *trajectory* segments are used.

3. CALCULATION OF A 4-3-4 JOINT TRAJECTORY

Since we are determining *N* joint trajectories in each trajectory segment, it is convenient to introduce a normalized time variable, $t \in [0, 1]$, which allows us to treat the equations of each trajectory segment for each joint angle in the same way, with time varying from *t* = 0 (initial time for all trajectory segments) to t = 1 (final time for all trajectory segments). Let us define the following variables:

t : normalized time variable, $t \in [0, 1]$
τ: real time in seconds
τ_i: real time at the end of the *i x h* trajectory segment
$t_i = \tau_i - \tau_{i-1}$: real time required to travel through the *i*th segment
$$t = \frac{\tau - \tau_{i-1}}{\tau_i - \tau_{i-1}};\tau \in [\tau_{i-1}, \tau_i];\ t \in [0,1]$$

The trajectory consists of the polynomial sequences, $h_{i(t)}$, which together form the I trajectory for joint *j.* The polynomial equations for each joint variable in each trajectory segment expressed in normalized time are:

(2) $$h_{1(t)} = a_{14}t^4 + a_{13}t^3 + a_{12}t^2 + a_{11}t + a_{10}$$ (1st segment)

(3)

(4) $h_{2(t)} = a_{23}t^3 + a_{22}t^2 + a_{21}t + a_{20}$ (2nd segment)

(5) $h_{n(t)} = a_{n4}t^4 + a_{n3}t^3 + a_{n2}t^2 + a_{n1}t + a_{n0}$ (last segment)

The subscript of each polynomial equation indicates the segment number, and *n* indicates the last trajectory segment. The unknown coefficient a_{ji} indicates the *i*th coefficient for the *j* trajectory segment of a joint trajectory. The boundary conditions that this set of joint trajectory segment polynomials must satisfy are:

1. Initial position = $\theta_0 = \theta_{(t_0)}$
2. Magnitude of initial velocity = v_0 (normally zero)
3. Magnitude of initial acceleration = a_Q (normally zero)
4. Lift-off position = $\theta_1 = \theta_{(t_1)}$
5. Continuity in position t_1, [that is, $\theta_{(t_1^-)} = \theta_{(t_1^+)}$]
6. Continuity in velocity at t_1, [that is, $v_{(t_1^-)} = v_{(t_1^+)}$]
7. Continuity in acceleration at t_1 [that is, $a_{(t_1^-)} = a_{(t_1^+)}$]
8. Set-down position = $\theta_2 = \theta_{(t_2)}$
9. Continuity in position at t_2 [that is, $\theta_{(t_2^-)} = \theta_{(t_2^+)}$]
10. Continuity in velocity at t_2 [that is, $v_{(t_2^-)} = v_{(t_2^+)}$]
11. Continuity in acceleration at t_2 [that is, $a_{(t_2^-)} = a_{(t_2^+)}$]
12. Final position = $\theta_f = \theta_{(t_f)}$
13. Magnitude of final velocity = v_f (normally zero)
14. Magnitude of final acceleration = a_f (normally zero)

The boundary conditions for the 4-3-4 joint trajectory are shown in Fig 1. The first and second derivatives of these polynomial equations with respect to real time τ can be written as:

(6) $v_{i(t)} = \frac{dh_{i(t)}}{d\tau} = \frac{dh_{i(t)}}{dt}\frac{dt}{d\tau} = \frac{1}{\tau_i - \tau_{i-1}}\frac{dh_{i(t)}}{dt} = \frac{1}{t_i}\frac{dh_{i(t)}}{dt} = \frac{1}{t_i}\dot{h}_i(t) \quad i = 1,2 \ldots n$

and

(7) $a_{i(t)} = \frac{d^2h_{i(t)}}{d\tau^2} = \frac{1}{(\tau_i - \tau_{i-1})^2}\frac{d^2h_{i(t)}}{dt^2} = \frac{1}{t_i^2}\frac{d^2h_{i(t)}}{dt^2} = \frac{1}{t_i^2}\ddot{h}_{i(t)}, i = 1,2 \ldots n$

For the first *trajectory segment, the* governing polynomial equation is of the fourth degree:

(7) $h_1(t) = a_{14}t^4 + a_{13}t^3 + a_{12}t^2 + a_{11}t + a_{10} \; t \in [0,1]$

From equations (5) and (6), its first two derivatives with respect to real time are:

(8) $v_1(t) = \frac{\dot{h}_{1(t)}}{t_1} = \frac{4a_{14}t^3 + 3a_{13}t^2 + 2a_{12}t^2 + 4a_1 t}{t_1}$

and

(9) $a_1(t) = \frac{\ddot{h}_{1(t)}}{t_1^2} = \frac{12a_{14}t^2 + 6a_{13}t + 2a_{12}t}{t_1^2}$

1. For $t = 0$ (at the initial position of the trajectory segment). Satisfying the boundary conditions at this point leads to:

(10) $a_{10} = h_1(0) = \theta_0$ (given) (11)

$v_0 = \frac{\dot{h}_1(0)}{t_1} = \left(\frac{4a_{14}t^4 + 3a_{13}t^4 + 2a_{12}t + a_{11}}{t_1^2}\right)_{t=0} = \frac{a_{11}}{t_1}$

which gives

$a_{11} = v_0 t_1$

and

(12) $a_0 = \frac{\ddot{h}_1(0)}{t_1} = \left(\frac{12a_{14}t^4 + 6a_{13}t + 2a_{12}}{t_1^2}\right)_{t=0} = \frac{2a_{12}}{t_1^2}$

which yields

$a_{12} = \frac{a_0 t_1^2}{2}$

Wit these unknowns determined, equation (12) can be rewritten as:

(13) $h_1(t) = a_{14}t^4 + a_{13}t^3 + \left(\frac{a_0 t_1^2}{2}\right)t^2 + (v_0 t_1)t + \theta_0, t \in [0,1]$

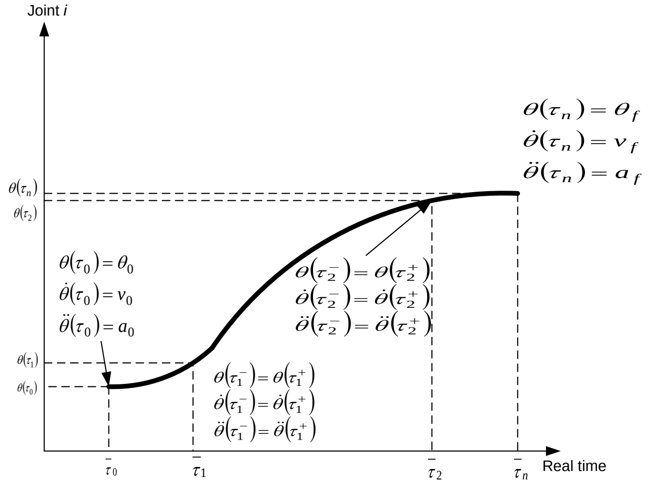

Fig. 1 Boundary conditions for a 4-3-4 joint trajectory

2. For $t = 1$ (at the final position of this trajectory segment). At this position, we relax the requirement that the interpolating polynomial must pass through the position exactly. We only require that the velocity and acceleration at this position have to be continuous with the velocity and acceleration, respectively, at the beginning of the next trajectory segment. The velocity and acceleration at this position are:

(14) $v_1(1) \equiv v_1 = \frac{\dot{h}_1(t)}{t_1} = \frac{4a_{14}+3a_{13}+a_0t_1^2+v_0t_1}{t_1}$

(15) $a(1) \equiv a = \frac{\ddot{h}_1(t)}{t_1^2} = \frac{12a_{14}+6a_{13}+a_0t_1^2}{t_1^2}$

For the secondary trajectory segment, the governing polynomial equation is of the third degree:

(16) $h_2(t) = a_{23}t^3 + a_{22}t^2 + a_{21}t + a_{20}\ ,t\ \in\ [0,1]$

For $t = 0$ (at the lift-off position). Using equations (5) and (6), the velocity and acceleration at this position are, respectively:

(17)

$$h_2(0) = a_{20} = \theta_2(0)$$

$$v_1 = \frac{\dot{h}_2(0)}{t_2} = \left(\frac{3a_{23}t^2 + 2a_{22}t + a_{21}}{t_2}\right)_{t=0} = \frac{a_{21}}{t_2}$$

which gives
$a_{21} = v_1t_2$
And
(19)

$$a_1 = \frac{\ddot{h}_2(0)}{t_2^2} = \left(\frac{6a_{23}t + 2a_{22}}{t_2^2}\right)_{t=0} = \frac{2a_{22}}{t_2^2}$$

which yields $a_{22} = \frac{a_1t_2^2}{2}$

Since the velocity and acceleration at this point must be continuous with the velocity and acceleration at the end of the previous trajectory segment, respectively, we have :

(20) $\frac{\dot{h}_2(0)}{t_2} = \frac{\dot{h}_1(1)}{t_1}\ and\ \frac{\ddot{h}_2(0)}{t_2^2} = \frac{\ddot{h}_1(1)}{t_1^2}$

which, respectively, leads to:

(21) $\left(\frac{3a_{23}t^2+2a_{22}t+a_{21}}{t_2}\right)_{t=0} = \left(\frac{4a_{14}t^3+3a_{13}t^2+2a_{12}t+a_{11}}{t_1}\right)_{t=1}$

or

(22) $\frac{-a_{21}}{t_2} + \frac{4a_{14}}{t_1} + \frac{3a_{13}}{t_1} + \frac{a_0t_1^2}{t_1} + \frac{v_0t_1}{t_1} = 0$

and

(23) $\left(\frac{6a_{23}t+2a_{22}}{t_2^2}\right)_{t=0} = \left(\frac{12a_{14}t^2+6a_{13}t+2a_{12}}{t_1^2}\right)_{t=1}$

or

(24) $\frac{-2a_{22}}{t_2^2} + \frac{12a_{14}}{t_1^2} + \frac{6a_{13}}{t_1^2} + \frac{a_0t_1^2}{t_1^2} = 0$

3. For $t = 1$ (at the set down position). Again the velocity and acceleration at this position must be continuous with the velocity and acceleration at the beginning of the next trajectory segment. The velocity and acceleration at this position are obtained, respectively, as :

(25) $h_2(1) = a_{23} + a_{22} + a_{21} + a_{20}$

(26) $v_2(1) = \frac{\dot{h}_2(1)}{t_2} = \frac{3a_{23}+2a_{22}+a_{21}}{t_2}$

and

(27) $a_2(1) = \frac{\ddot{h}_2(1)}{t_2^2} = \frac{6a_{23}+2a_{22}}{t_2^2}$

For the last trajectory segment, the governing polynomial equation is of the fourth degree:

(28) $h_n(t) = a_{n4}t^4 + a_{n3}t^3 + a_{n2}t^2 + a_{n1}t + a_{n0}\,, t \in [0,1]$

If we substitute $\bar{t} = t - 1$ into t in the above equation, we have shifted the normalized time t from $t \in [0,1]$ to $\bar{t} \in [-1,0]$. Then equation (28) becomes:

(29) $h_n(\bar{t}) = a_{n4}\bar{t}^4 + a_{n3}\bar{t}^3 + a_{n2}\bar{t}^2 + a_{n1}\bar{t} + a_{n0}, t \in [-1,0]$

Using equations (5) and (6), its first and second derivatives with respect to real time are:

(30) $v_n(\bar{t}) = \frac{\dot{h}_n(\bar{t})}{t_n} = \frac{4a_{n4}\bar{t}^3 + 3a_{n3}\bar{t}^2 + 2a_{n2}\bar{t} + a_{n1}}{t_n}$

and

(31) $a_n(\bar{t}) = \frac{\ddot{h}_n(\bar{t})}{t_n^2} = \frac{12a_{n4}\bar{t}^2 + 6a_{n3}\bar{t} + 2a_{n2}}{t_n^2}$

1. For $\bar{t} = 0$ (at the final position of segment). Satisfying the boundary conditions at the final position of the trajectory, we have:

(32) $h_n(0) = a_n(0) = \theta_f$

(33) $v_f = \frac{\dot{h}_n(0)}{t_n} = \frac{a_{n1}}{t_n}$

which gives

$a_{n1} = v_f t_n$

and

(34) $a_f = \frac{\ddot{h}_n(0)}{t_n^2} = \frac{2a_{n2}}{t_n^2}$

which yields

$a_{n2} = \frac{a_f t_n^2}{2}$

2. For $\bar{t} = -1$ (at the starting position of this trajectory segment). Satisfying the boundary conditions at this position, we have, at the set-down position:

(35) $h_n(-1) = a_{n4} - a_{n3} + \frac{a_f t_n^2}{2} - v_f t_n + \theta_f = \theta_2(1)$

and

(36) $\frac{\dot{h}_n(-1)}{t_n} = \left(\frac{4a_{n4}\bar{t}^3 + 3a_{n3}\bar{t}^2 + 2a_{n2}\bar{t} + a_{n1}}{t_n}\right)_{\bar{t}=-1} = \frac{-4a_{n4} + 3a_{n3} - a_f t_n^2 + a_{n1}}{t_n}$

and

(37) $\frac{\ddot{h}_n(-1)}{t_n^2} = \left(\frac{12a_{n4}\bar{t}^2 + 6a_{n3}\bar{t} + 2a_{n2}}{t_n^2}\right)_{\bar{t}=-1} = \frac{12a_{n4} - 6a_{n3} + a_f t_n^2}{t_n^2}$

The velocity and acceleration continuity conditions at this set-down point are:

(38) $\frac{\dot{h}_2(-1)}{t_2} = \frac{\dot{h}_n(-1)}{t_n}$ and $\frac{\ddot{h}_2(-1)}{t_2^2} = \frac{\ddot{h}_n(-1)}{t_n^2}$

or

(39) $\frac{4a_{n4} - 3a_{n3} + a_f t_n^2 - v_f t_n}{t_n} + \frac{3a_{23}}{t_2} + \frac{2a_{22}}{t_2} + \frac{a_{21}}{t_2} = 0$

and

(40) $\frac{-12a_{n4} + 6a_{n3} - a_f t_n^2}{t_n^2} + \frac{6a_{23}}{t_2^2} + \frac{2a_{22}}{t_2^2} = 0$

The difference of joint angles between successive trajectory segments can be found to be:

(41) $\delta_1 = \theta_1 - \theta_0 = h_1(1) - h_1(0) = a_{14} + a_{13} + \frac{a_0 t_1^2}{2} - v_0 t_1$

(42) $\delta_2 = \theta_2 - \theta_1 = h_2(1) - h_2(0) = a_{23} + a_{22} + a_{21}$

and

(43) $\delta_n = \theta_f - \theta_2 = h_n(0) - h_n(-1) = -a_{n4} + a_{n3} - \frac{a_f t_n^2}{2} + v_f t_n$

All the unknown coefficients of the trajectory polynomial equations can be determined by simultaneously solving equations (41),(22), (24), (42), (39), (40) and (43). Rewriting them in the matrix vector notation, we have:

(44) $y = Cx$

where

(45) $y = \left(\delta_1 - \frac{a_0 t_1^2}{2} - v_0 t_1, a_0 t_1 - v_0, -a_0, \delta_2, -a_f t_n + v_f, a_f, \delta_n - \frac{a_f t_n^2}{2} - v_f t_n\right)^T$

(46) $$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3/t_1 & 4/t_1 & -1/t_1 & 0 & 0 & 0 & 0 \\ 6/t_1^2 & 12/t_1^2 & 0 & -2/t_2^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1/t_2 & 2/t_2 & 3/t_2 & -3/t_n & 4/t_n \\ 0 & 0 & 0 & 2/t_2^2 & 6/t_2^2 & 6/t_n^2 & -12/t_n^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

and

(47) $x = (a_{13}, a_{14}, a_{21}, a_{22}, a_{23}, a_{n3}, a_{n4})^T$

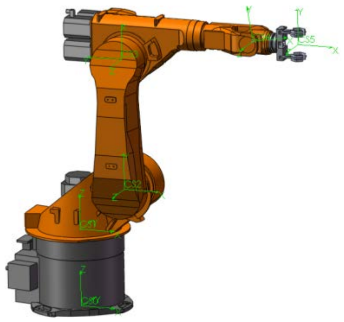

Fig. 2 KUKA manipulator

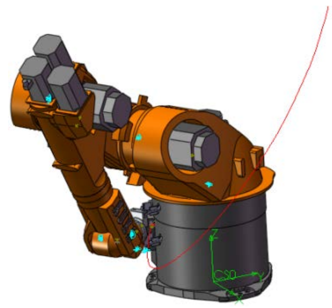

Fig. 3 The joint trajectory of the gripper

Then the planning of the joint trajectory (for each joint) reduces to solving the matrix vector equation (44):

(48) $y_i = \sum_{j=1}^{7} c_{ij} x_j$

or

(49) $x = C^{-1} y$

The structure of matrix C makes it easy to compute the unknown coefficients and if the inverse of C always exists if the time intervals t_i, $i = 1,2,..n$ are positive values. Solving Eq. (43-49), we obtain all the coefficients for the polynomial equations for the joint trajectory segments for joint j.

Since we made a change in normalized time to run from [0, 1] to [— 1,0] for the last trajectory segment, after obtaining the coefficients a_{ni} from the above matrix equation, we need to reconvert the normalized time back to [0, 1]. This can be accomplished by substituting $t = \bar{t} + 1$ into $\bar{t}$ in equation (29). Thus we obtain:

$$h_n(t) = a_{n4}t^4 + (-4a_{n4} + a_{n3})t^3 + (6a_{n4} - 3a_{n3} + a_{n2})t^2 +$$
$$+(-4a_{n4} + 3a_{n3} - 2a_{n2} + a_{n1})t$$

(50) $+(a_{n4} - a_{n3} + a_{n2} - a_{n1} + a_{n0})$

In figures 2 and 3 we have the joint trajectory for a gripper which it is attached to KUKA manipulator.

4. CONCLUDING REMARKS

Two major approaches for trajectory planning have been discussed: the joint-interpolated approach and the Cartesian space approach. The joint-interpolated approach plans polynomial sequences that yield smooth joint trajectory. In order to yield faster computation and less extraneous motion, lower-degree polynomial sequences are preferred. The joint trajectory is split into several trajectory segments and each trajectory segment is splined by a low-degree polynomial. In particular, 4-3-4 has been discussed. Because servoing is done in the joint-variable space while a path is specified in Cartesian coordinates, the most common approach is to plan the straight-line path in the joint-variable space using low-degree polynomials to approximate the path. These techniques represent a shift away from the real-time planning objective to an off-line planning phase. In essence, this decomposes the control of robot manipulators into off-line motion planning followed by on-line tracking control.

REFERENCES

1. T. Binford *et al., Exploratory Studies of Computer Integrated Assembly Systems,* NSF Progress Reports, Stanford Artificial Intelligence Laboratory, Stanford University, Stanford, CA, September 1974, November 1975, July 1976(AIM 285).

2. C. Rosen *et al., Machine Intelligence Research Applied to Industrial Automation,* NSF Report, Grant APR75-13074, Stanford Research Institute, Stanford, CA, November 1976.

3. P. M. Will and D. D. Grossman, "An Experimental Systemfor Computer Controlled Mechanical Assembly," *ZEEETrans. Computers* (2-24, No. 9, 879-888 (1975).

4. R. Finkel, R. Taylor, R. Bolles, R. Paul, and J. Feldman, *AL, A Programming System for Automation* St anford Artificial Intelligence Laboratory Memo AIM-177, Stanford Computer Science Report STAN-(3-74-456, Stanford University, Stanford, CA, November 1974.

5. R. H. Taylor, "The Synthesis of Manipulator Control Programs from Task-Level Specifications," Ph.D. Thesis, Stanford Artificial Intelligence Laboratory Memo AIM-282, Stanford Computer Science Report STAN-CS-76-560, Stanford University, Stanford, CA, July 1976.