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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF n-TH ORDER DIFFERENTIAL EQUATIONS WITH MAXIMA

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Abstract: *This paper deals with an n-th order forced differential equations with „maxima" where n is at least equal to 2. These types of equations have been proposed in the connection with the theory of optimal control to different physical systems. In the theory of automatic control of various mechanical technological systems it often occurs that the law of regulation depends on the maximum values of some regulated state parameters over certain past time intervals. In the literature there are only some few results about oscillation properties of differential equations with "maxima", which require deeper investigation of this property. In the paper various types of sufficient conditions for approaching zero of the iterated derivative of every oscillatory solution of the given equation are obtained. Also an estimate of the growth of the solutions is given.*

AMS (MOS) Subject Classification : *34K15, 34K11*
Key Words: *differential equations with "maxima", damped oscillation, bounded solutions*

1 INTRODUCTION

The study of differential equations with "maxima" starts with the papers of A. Magomedov [8], [9], where the first ideas for linear differential equations with "maxima" have been proposed in the connection with the theory of optimal control to different physical systems. In the theory of automatic control of various mechanical technological systems it often occurs that the law of regulation depends on the maximum values of some regulated state parameters over certain time intervals ([14]). At the same time higher order differential equations have been applied to mechanics, vibration theory, engineering and technology ([12], [13]). This proves the necessity of investigation various properties of solutions of higher order differential equations with "maxima". Note some initial oscillation results for differential equations with "maxima" are obtain in [1] - [7], [10].

2. PRELIMINARY NOTES

Consider the following scalar differential equation with "maxima"

$$D_r^{(n)}x(t) + a(t)F\left(\max_{s\in I(t)} x(s)\right) = b(t) \qquad \text{for } t \ge \alpha, \tag{1.1}$$

where $x \in \mathbb{R}$, $n \ge 2$ is an integer, $I(t)=[\sigma(t),\tau(t)]$, the functions $\sigma, \tau \in C(J, R)$ are such that $\sigma(t)\le\tau(t)\le t$, and $J=[\alpha,+\infty)$, $\alpha\ge 0$ is a constant, $a, b\colon J \to R$, $F\colon R \to R$ and

$$D_r^{(0)}x(t) = r_0(t)x(t), \quad D_r^{(i)}x(t) = r_i(t)(D_r^{(i-1)}x(t))', \quad i = 1,\dots,n$$

where $r_i\colon J \to (0,+\infty)$, $i = 0,1,\dots,n$.

Remark 1. *If there exists a point $\xi \ge \alpha$ such that $\sigma(\xi) = \tau(\xi)$ then we have $\max_{s\in I(\xi)} x(s) = x(\tau(\xi))$.*

We introduce the following set of conditions:

H1. The functions $\sigma, \tau \in C(J, \mathbb{R})$, $\sigma(t) \le \tau(t) \le t$, $\lim_{t\to+\infty} \sigma(t) = +\infty$ and there exists a positive constant $h < \infty$ such that

$$\sup_{t\ge\alpha}\big(t-\sigma(t)\big) \le h. \tag{1.2}$$

H2. The functions $a, b \in C(J, \mathbb{R})$.

H3. The functions $r_k \in C(J,(0,+\infty))$, $k = 0,1,\dots n-1$ and $r_n(t) \equiv 1$.

H4. The function $F \in C(R,R)$ and there exists a constant $K > 0$ such that $|F(x)| \le K|x|$ for $x \in R$.

Remark 2. *If condition H1 is satisfied then for any $t \ge \alpha$ the inequality*

$$h \ge \sup_{t\ge\alpha}\big(t-\sigma(t)\big) \ge t-\sigma(t) \ge \alpha - \sigma(t)$$

holds, i.e. $\sigma(t) \ge \alpha - h$, i.e. the inclusion $I(t) \subset [\alpha - h, \infty)$ is valid for $t \ge \alpha$.

Let $T \in J$ be a given number.

Definition 1. The function $x(t)$ is called to be a *solution* of differential equation with "maxima" (1.1) on the interval $[T,+\infty)$, $T\geq\alpha$ if it is defined for $t\geq T-h$ and it satisfies (1.1) for $t\geq T$.

Definition 2. The solution $x(t)$ of differential equation with "maxima" (1.1) in the interval $[T,+\infty)$ is said to be:

1. *a proper solution* if there exists a number $T_x\geq T$ such that
$$\sup\{|x(t)|: t\geq T_1\}>0 \quad \text{for all} \quad T_1\geq T_x.c$$
2. *non-oscillator solution*, if it is a proper solution and it is either positive or negative for $t\geq T_x$.
3. *oscillatory solution*, if it is a proper solution and there are infinite number of points on $[T,+\infty)$ at which the solution changes its sign.

Equation (1.1) is said to be *oscillatory*, if all proper solutions of equation (1.1) are oscillatory.

Remark 3. *Note that a proper solution of equation* (1.1) *is* $oscillatory$ *if it has arbitrarily large zeros; otherwise it is* $nonoscillatory$.

The main purpose of this paper is obtaining sufficient conditions for validity of
$$\lim_{t\to+\infty} D_r^{(i)}x(t)=0, \quad i=0,1,\dots,n-1,$$
where $x(t)$ is a solution of n-th order scalar differential equation with "maxima" (1.1).

In the paper four main cases are studied:

– the oscillatory solution $x(t)$ of equation (1.1) satisfies $x(t)=O(\mu(t))$ as $t\to+\infty$, where $\mu(t)$ is a positive nondecreasing continuous function on J (Theorem 1 and Theorem 2);
– the oscillatory solution $x(t)$ of equation (1.1) satisfies $x(t)=O(\lambda(t))$ as $t\to+\infty$, where $\lambda(t)$ is a positive nonincreasing continuous function on J (Theorem 5 and Theorem 6);
– the solution $x(t)$ of (1.1) is oscillatory (Theorem 4);
– the solution $x(t)$ of (1.1) is bounded (Theorem 7);

are considered.

Also the growth of the solution of equation (1.1) is estimated (Theorem 3).

The main results generalize and extend results obtained by Philos ([11]) where $\sigma(t)=\tau(t)=g(t)$ and the maximum function reduces to $\max_{s\in I(t)} x(s)=x(g(t))$.

3 Main results

Theorem 1. *Let the following conditions be fulfilled:*
1. *Conditions H1 - H4 are satisfied.*
2. *The inequality*
$$\int^{\infty}\frac{1}{r_1(s_1)}\int_{s_1}^{\infty}\frac{1}{r_2(s_2)}\dots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}|b(s)|\,ds\dots ds_1<+\infty$$
holds.
3. *There exists a nondecreasing function* $\mu\in C(J,(0,+\infty))$ *such that*
$$\int^{\infty}\frac{1}{r_1(s_1)}\int_{s_1}^{\infty}\frac{1}{r_2(s_2)}\dots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}\mu(\tau(s))|a(s)|\,ds\dots ds_1<+\infty.$$

Then every oscillatory solution $x(t)$ *of equation* (1.1) *with* $x(t)=O(\mu(t))$ as $t\to+\infty$ *satisfies*
$$\lim_{t\to+\infty} D_r^{(i)}x(t)=0, \quad i=0,1,\dots,n-1. \qquad (1.3)$$

P r o o f: Let $x(t)$ be an oscillatory solution of equation (1.1) in the interval $[T_0,+\infty)\subseteq J$ for which $x(t)=O(\mu(t))$ as $t\to+\infty$. From condition H1 it follows that there exists $T\geq T_0$ such that $\tau(t)\geq\sigma(t)\geq T_0$ for $t\geq T$.

For any integer *k, 0≤k≤n-1* and each t≥T we have

$$|D_r^{(k)}x(t)| \le$$
$$\le \begin{cases} \int_t^{\infty} |D_r^{(n)}x(s)|\,ds, & \text{for } k = n-1 \\ \int_t^{\infty} \frac{1}{r_{k+1}(s_{k+1})} \dots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |D_r^{(n)}x(s)|\,ds \dots ds_{k+1}, & \text{for } k < n-1. \end{cases} \quad (1.4)$$

Indeed, since x is oscillatory, all functions $D_r^{(i)}x,\ i = k, k+1, \dots, n-1$ are also oscillatory and for any fixed $t \ge T$ we can choose $\tau_i,\ i = k, k+1, \dots, n-1$ such that $t \le \tau_k \le \tau_{k+1} \le \dots \le \tau_{n-1}$ and

$$D_r^{(i)}x(\tau_i) = 0, \quad i = k, k+1, \dots, n-1.$$

Therefore, after several times integration we obtain

$$(-1)^{n-k} D_r^{(k)}x(t) =$$
$$= \begin{cases} \int_t^{\tau_{n-1}} D_r^{(n)}x(s)\,ds, & \text{for } k = n-1 \\ \int_t^{\tau_k} \frac{1}{r_{k+1}(s_{k+1})} \dots \int_{s_{n-2}}^{\tau_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\tau_{n-1}} D_r^{(n)}x(s)\,ds \dots ds_{k+1}, & \text{for } k < n-1. \end{cases}$$

Since $t \le \tau_k \le \tau_{k+1} \le \dots \le \tau_{n-1}$, we get the inequalities

$$|D_r^{(k)}x(t)| \le$$
$$\le \begin{cases} \int_t^{\tau_{n-1}} |D_r^{(n)}x(s)|\,ds, & \text{for } k = n-1 \\ \int_t^{\tau_k} \frac{1}{r_{k+1}(s_{k+1})} \dots \int_{s_{n-2}}^{\tau_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\tau_{n-1}} |D_r^{(n)}x(s)|\,ds \dots ds_{k+1}, & \text{for } k < n-1. \end{cases}$$

The above inequality implies the validity of (1.4).

Since $|x(t)| = O(\mu(t))$ as $t \to +\infty$ and $\mu(t)$ is a nondecreasing function, there exists a constant $A > 0$ such that

$$|\max_{s \in I(t)} x(s)| \le \max_{s \in I(t)} |x(s)| \le A \max_{s \in I(t)} \mu(s) = A\mu(\tau(t))), \quad t \ge T. \quad (1.5)$$

According to condition H4 and inequality (1.5) we obtain $|F(\max_{s \in I(t)} x(s))| \le KA\mu(\tau(t))$. Substitute it in the equation (1.1) and get

$$|D_r^{(n)}x(t)| \le KA\mu(\tau(t))|a(t)| + |b(t)|, \quad \text{for } t \ge T. \quad (1.6)$$

From inequalities (1.4) and (1.6) we have for any integer $k,\ 0 \le k \le n-1$, and each $t \ge T$

$$|D_r^{(k)}x(t)| \le$$
$$\le \begin{cases} KA\int_t^{\infty} \mu(\tau(s))|a(s)|\,ds + \int_t^{\infty} |b(s)|\,ds, & \text{for } k = n-1 \\ KA\int_t^{\infty} \frac{1}{r_{k+1}(s_{k+1})} \dots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} \mu(\tau(s))|a(s)|\,ds \dots ds_{k+1} & \\ \quad + \int_t^{\infty} \frac{1}{r_{k+1}(s_{k+1})} \dots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)|\,ds \dots ds_{k+1}, & \text{for } k < n-1 \end{cases} \quad (1.7)$$

Inequality (1.7) and conditions 2 and 3 of Theorem 1 imply the validity of (1.3).

□

In the case when a derivative of the solution is oscillatory instead of the solution we obtain the following general result:

Theorem 2. *Let the following conditions be fulfilled:*
1. *Conditions H1 - H4 are satisfied.*
2. *For some* $k: 0 \le k \le n-1$ *the inequality*

$$\int^{\infty} |b(s)|\, ds < \infty, \qquad if \quad k = n-1$$

or

$$\int^{\infty} \frac{1}{r_{k+1}(s_{k+1})} \dots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)|\, ds \dots ds_{k+1} < +\infty, \quad if \quad k < n-1$$

holds.

3. *There exists a nondecreasing function* $\mu \in C(J,(0,+\infty))$ *such that the inequality*

$$\int^{\infty} \mu(\tau(s))\,|a(s)|\, ds < +\infty, \qquad if \quad k = n-1$$

or

$$\int^{\infty} \frac{1}{r_{k+1}(s_{k+1})} \dots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} \mu(\tau(s))\,|a(s)|\, ds \dots ds_{k+1} < +\infty, \quad if \quad k < n-1$$

holds, where the integer k *is defined in the condition 2.*

Then every solution $x(t)$ *of differential equation with “maxima”* (1.1) *such that* $x(t) = O(\mu(t))$ *as* $t \to +\infty$ *and its* r *-derivative* $D_r^{(k)} x$ *is oscillatory, satisfies*

$$\lim_{t \to +\infty} D_r^{(i)} x(t) = 0, \quad i = k, k+1, \dots, n-1.$$

The proof of Theorem 2 is similar to the proof of Theorem 1 and we omit it.

As partial cases of the above Theorems we obtain the following sufficient conditions for asymptotic convergence of the derivatives of the oscillatory solutions of the differential equation with “maxima” (1.1):

Corollary 1. *Let the following conditions be fulfilled:*
1. *Conditions H1 - H4 are satisfied and*

$$\int^{\infty} |b(s)|\, ds < +\infty, \qquad \int^{\infty} \frac{1}{r_i(s)}\, ds < +\infty, \quad i = 1, \dots, n-1.$$

2. *There exists a nondecreasing function* $\mu \in C(J,(0,+\infty))$ *such that*

$$\int^{\infty} \mu(\tau(s))\,|a(s)|\, ds < +\infty.$$

Then every oscillatory solution $x(t)$ *of differential equation with “maxima”* (1.1) *such that* $x(t) = O(\mu(t))$ *as* $t \to +\infty$ *satisfies* $\lim_{t \to +\infty} D_r^{(i)} x(t) = 0$, $i = 0, 1, \dots, n-1.$

Corollary 2. *Let the following conditions be fulfilled:*
1. *Conditions H1 - H4 are satisfied.*
2. *There exist constants* $T \ge \alpha$ *and* $c > 0$ *such that* $r_i(t) \ge c$ *for* $i = 0, 1, \dots, n-1.\ t \ge T,$
3. *The inequality*

$$\int^{\infty} s^{n-1}\,|b(s)|\, ds < +\infty$$

holds.

4. *There exists a nondecreasing function* $\mu \in C(J,(0,+\infty))$ *such that*

$$\int^{\infty} s^{n-1}\mu(\tau(s))\,|\,a(s)\,|\,ds < +\infty.$$

Then every oscillatory solution $x(t)$ *of differential equation with "maxima"* (1.1) *such that* $x(t)O(\mu(t))$ *as* $t \to +\infty$ *satisfies* $\lim_{t\to+\infty} D_r^{(i)}x(t) = 0,\ \ i = 0,1,\ldots,n-1.$

Define functions $R_k : J \to (0,+\infty),\ k = 0,1,\ldots,n-1$ by the equalities:

$$R_k(t) = \begin{cases} 1, & for\ k = 0 \\ \int_\alpha^t \frac{1}{r_1(s_1)}\int_\alpha^{s_1}\frac{1}{r_2(s_2)}\ldots\int_\alpha^{s_{k-1}}\frac{1}{r_k(s_k)}ds_k\ldots ds_1, & for\ 0 < k \le n-1 \end{cases}$$

and set $R(t) = R_{n-1}(t).$

For $k = 0,1,\ldots,n-1$ and $t \ge T$ we define the functions

$$G_k(t,T) = \begin{cases} 1, & for\ k = 0 \\ \int_T^t \frac{1}{r_1(s_1)}\int_T^{s_1}\frac{1}{r_2(s_2)}\ldots\int_T^{s_{k-1}}\frac{1}{r_k(s_k)}ds_k\ldots ds_1, & for\ 0 < k \le n-1 \end{cases}$$

Note that $G_k(t,\alpha) = R_k(t).$

Introduce the following conditions:

H5. $\liminf_{t\to+\infty} r_0(t) > 0.$

H6. $\lim_{t\to+\infty}\frac{R_k(t)}{R(t)} < +\infty, k = 0,1,\ldots,n-1.$

H7. $\int^{\infty} R(\tau(t))\,|\,a(t)\,|\,dt < +\infty.$

H8. $\int^{\infty} |\,b(t)\,|\,dt < +\infty.$

Theorem 3. *Let conditions H1 - H8 be fulfilled.*

Then every solution $x(t)$ *of differential equation with "maxima"* (1.1)*, which is defined for* $t \ge T_0 \ge \alpha$ *satisfies*

$$x(t) = O(R(t)) \quad as\ \ t \to +\infty.$$

P r o o f: Let $x(t), t \ge t_0 \ge \alpha$ be a solution of equation (1.1). According to condition H1 there exists $t \ge T_0$ such that $\tau(t) \ge \sigma(t) \ge T_0,\ \ t \ge T.$

From equation (1.1) we obtain for $t \ge T$

$$D_r^{(0)}x(t) = \sum_{k=0}^{n-1} D_r^{(k)}x(T)G_k(t,T) + \int_T^s \frac{1}{r_1(s_1)}\int_T^{s_1}\frac{1}{r_2(s_2)}\ldots\int_T^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_T^{s_{n-1}} D_r^{(n)}x(s)ds\ldots ds_1,$$

and therefore

$$|\,r_0(t)x(t)\,| \le \sum_{k=0}^{n-1}|D_r^{(k)}x(T)\,|\,R_k(t) + R(t)\int_T^t |D_r^{(n)}x(s)\,|\,ds, \quad t \ge T. \quad (1.8)$$

It follows from conditions H5 and H6 that there exists $c > 0$ such that

$$r_0(t) \ge c, \quad R_k(t) \le cR(t) \quad for \;\; t \ge T \quad and \quad \sum_{k=0}^{n-1} |D_r^{(k)} x(T)| \le c.$$

From inequality (1.8) we get

$$|\,x(t)\,| \le cR(t) + \frac{1}{c} R(t) \int_T^t |D_r^{(n)} x(s)\,|\,ds, \quad t \ge T. \tag{1.9}$$

Inequalities (1.9) and

$$|\max_{s\in I(t)} x(s)\,| \le \max_{s\in I(t)} |\,x(s)\,| \le c_0 \max_{s\in I(t)} R(s) = c_0 R(\tau(t)),$$

where $c_0 > 0$ imply that

$$|\,x(t)\,| \le R(t)\left\{ c + \frac{1}{c} \int_T^t \left[|b(s)| + |a(s)| K \left| \max_{\xi\in I(s)} x(\xi) \right| \right] ds \right\}$$

$$\le R(t)\left\{ c + \frac{1}{c} \int_T^t \left[|b(s)| + |a(s)| K c_0 R(\tau(s)) \right] ds \right\} \tag{1.10}$$

$$\le R(t)\left\{ c + \frac{1}{c} \left[\int_T^\infty |b(s)| ds + \int_T^\infty |a(s)| K c_0 R(\tau(s)) ds \right] \right\}.$$

It follows from (1.10) and conditions H7 and H8 that

$$|\,x(t)\,| \le MR(t) \quad for \quad t \ge T \;\; and \;\; some \;\; M > 0.$$

This means that $x(t) = O(R(t))$ as $t \to +\infty$.

Theorem 4. *Assume that conditions H1 - H6 and*

$$\int^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \ldots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |\,b(s)\,|\,ds \ldots ds_1 < +\infty$$

$$\int^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \ldots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} R(\tau(s))\,|\,a(s)\,|\,ds \ldots ds_1 < +\infty.$$

Then every oscillatory solution of differential equation with "maxima" (1.1) *satisfies* $\lim_{t\to+\infty} D_r^{(i)} x(t) = 0 \quad for \;\; i = 0,1,\ldots,n-1.$

P r o o f: Theorem 4 is a corollary of Theorem 1 since the function $\mu(t) \equiv R(t)$ is nondecreasing.

Now we will consider the case when the function $\mu(t)$ in Theorem 1 is a nonincreasing one.

Theorem 5. *Let the following conditions be fulfilled:*

1. *Conditions H1 - H4 are satisfied.*
2. *The inequality*

$$\int^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \ldots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |\,b(s)\,|\,ds \ldots ds_1 < +\infty$$

holds.

1. *There exists a nonincreasing function* $\lambda \in C(J,(0,+\infty))$ *such that*

$$\int^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \ldots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \lambda(\sigma(s))\,|\,a(s)\,|\,ds \ldots ds_1 < +\infty.$$

Then every oscillatory solution $x(t)$ *of differential equation with "maxima"* (1.1) *such that* $x(t) = O(\lambda(t))$ *as* $t \to +\infty$

satisfies $\lim\limits_{t\to+\infty} D_r^{(i)}x(t)=0$ *for* $i=0,1,\dots,n-1.$

P r o o f: Let $x(t)$ be an oscillatory solution of equation (1.1) in the interval $[T_0,+\infty)\subseteq J$ for which $x(t)=O(\lambda(t))$ as $t\to+\infty.$ Since the function $\lambda(t)$ is nonincreasing, there exists a constant $B>0$ such that

$$|\max_{s\in I(t)} x(s)|\le \max_{s\in I(t)}|x(s)|\le B\max_{s\in I(t)}\lambda(s)=B\lambda(\sigma(t)),\quad t\ge T. \qquad (1.11)$$

Then from condition H4 and inequality (1.11) we get

$$|F(\max_{s\in I(t)} x(s))|\le K|\max_{s\in I(t)} x(s)|\le KB\lambda(\sigma(t)),\quad t\ge T \qquad (1.12)$$

From equation (1.1) and inequality (1.12) we obtain

$$|D_r^{(n)}x(t)|\le KB\lambda(\sigma(t))\,|a(t)|+b(t),\quad t\ge T. \qquad (1.13)$$

As in the proof of Theorem 1 we set up the validity of inequality (1.4). From inequalities (1.4) and (1.13) it follows

$$|D_r^{(k)}x(t)|\le$$

$$\le\begin{cases} KB\int_t^\infty \lambda(\sigma(s))\,|a(s)|\,ds+\int_t^\infty |b(s)|\,ds, & for\ \ k=n-1 \\ KB\int\limits_t^\infty \frac{1}{r_{k+1}(s_{k+1})}\dots\int\limits_{s_{n-2}}^\infty \frac{1}{r_{k+1}(s_{k+1})}\int\limits_{s_{n-1}}^\infty \lambda(\sigma(s))\,|a(s)|\,ds\dots ds_{k+1} & \\ \qquad +\int\limits_t^\infty \frac{1}{r_{k+1}(s_{k+1})}\dots\int\limits_{s_{n-2}}^\infty |b(s)|\,ds\dots ds_{k+1}, & for\ \ k<n-1. \end{cases} \qquad (1.14)$$

for any integer $k,\ 0\le k\le n-1$ and each $t\ge T.$

The inequality (1.14) and conditions 2 and 3 of Theorem 5 proof the claim of the Theorem.

In the case when a derivative of the solution is oscillatory instead of the solution and the existing function $\lambda(t)$ is nonicreasing we obtain the following result:

Theorem 6. *Let the following conditions be fulfilled:*
1. *Conditions H1 - H4 are satisfied.*
2. *For some* $k:\ 0\le k\le n-1$ *the inequality*

$$\int^\infty |b(s)|\,ds<\infty, \qquad if \quad k=n-1$$

or

$$\int\limits^\infty \frac{1}{r_{k+1}(s_{k+1})}\dots\int\limits_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int\limits_{s_{n-1}}^\infty |b(s)|\,ds\dots ds_{k+1}<+\infty, \quad if \quad k<n-1$$

holds.

3. *There exists a nonincreasing function* $\lambda\in C(J,(0,+\infty))$ *such that the inequality*

$$\int^\infty \lambda(\sigma(s))\,|a(s)|\,ds<+\infty, \qquad if \quad k=n-1$$

or

$$\int\limits^\infty \frac{1}{r_{k+1}(s_{k+1})}\dots\int\limits_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int\limits_{s_{n-1}}^\infty \lambda(\sigma(s))\,|a(s)|\,ds\dots ds_{k+1}<+\infty, \quad if \quad k<n-1$$

holds, where the integer k *is defined in the condition 2.*

Then every solution $x(t)$ *of differential equation with “maxima”* (1.1) *such that* $x(t)=O(\lambda(t))$ *as* $t\to+\infty$ *and its* r *-derivative* $D_r^{(k)}x$ *is oscillatory, satisfies* $\lim\limits_{t\to+\infty} D_r^{(i)}x(t)=0$ *for* $i=0,1,\dots,n-1.$

As a partial case of Theorem 5 we obtain the following sufficient conditions:

Corollary 3. *Let the following conditions be fulfilled:*

4. *Condition 1 of Corollary 1 is satisfied.*
5. *There exists a nonincreasing function* $\lambda \in C(J,(0,+\infty))$ *such that*

$$\int^{\infty} \lambda(\sigma(s))\,|\,a(s)\,|\,ds < +\infty.$$

Then every oscillatory solution $x(t)$ *of differential equation with “maxima”* (1.1) *such that* $x(t) = O(\lambda(t))$ *as* $t \to +\infty$ *satisfies* $\lim_{t\to+\infty} D_r^{(i)} x(t) = 0$ *for* $i = 0,1,\ldots,n-1.$

Corollary 4. *Let the following conditions be fulfilled:*

1. *Conditions 1, 2, and 3 of Corollary 2 are satisfied.*
2. *There exists a nonincreasing function such that*

$$\int^{\infty} s^{n-1} \lambda(\sigma(s))\,|\,a(s)\,|\,ds < +\infty. \tag{1.15}$$

Then every oscillatory solution $x(t)$ *of differential equation with “maxima”* (1.1) *such that* $x(t) = O(\lambda(t))$ *as* $t \to +\infty$ *satisfies* $\lim_{t\to+\infty} D_r^{(i)} x(t) = 0$ *for* $i = 0,1,\ldots,n-1.$

Remark 4. *In the case, when* $\sigma(t) = \tau(t) = g(t),\ \max_{s\in I(t)} x(s) = x(g(t))$ *the assumption that* $\mu(t)$ *and* $\lambda(t)$ *are monotonic is unnecessary. This case is considered in [11].*

The following theorem is corollary of Theorem 5 (with $\lambda \equiv 1$) and it concerns the bounded solutions of equation (1.1).

Theorem 7. *Let the conditions 1 and 2 of Theorem 1 be fulfilled and*

$$\int^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |\,a(s)\,|\,ds \ldots ds_1 < +\infty.$$

Then every bounded solution $x(t)$ *of differential equation with “maxima”* (1.1) *satisfies*

$$\lim_{t\to+\infty} D_r^{(i)} x(t) = 0 \quad \textit{for} \ \ i = 0,1,\ldots,n-1.$$

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